Lesson 21

DIFFERENT SEQUENCES

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and number relationships
Assessment Standard

- Investigate number patterns including but not limited to those where there is a constant difference and hence.
- Make conjectures and generalisations,
- Provide explanations and justifications and attempt to prove conjectures.

Overview

In this lesson you will:

- Look at unusual sequences where there is no first or second difference.
- If possible find the nth term.
- Prove conjectures.

LC DVD

Lesson

Example 1

What will the next three terms be in the sequence 2; 6; 18; 54; …

	T_1	T_2	T_3	T_4		
$\Rightarrow$	2 ;	6 ;	18 ;	54 ;	…	
	$\frac{1}{3}\times 6$;	1×6 ;	3×6 ;	9×6 ;		
=	$3^{-1}\times 6$;	$3^0\times 6$;	$3^1\times 6$;	$3^2\times 6$;	$3^3\times 6$;	$3^4\times 6$

So the next 3 terms will be: 162; 486; 1458

and $T_n = 3^{n-2}\times 6 = 3^n.3^{-2}.3.2 = 2.3^{n-1}$

Alternatively:

	T_1	T_2	T_3	T_4		
$\Rightarrow$	2 ;	6 ;	18 ;	54;		
=	2×1 ;	2×3 ;	2×9 ;	2×27		
=	2×3^0 ;	2×3^1 ;	2×3^2 ;	2×3^3 ;	2×3^4 ;	2×3^5 ;

…

So the next 3 terms will be: $2\times 3^4 = 162$

$2\times 3^5 = 486$

$2\times 3^6 = 1458$

Example 2

Again they all are multiples of 6:

	T_1	T_2	T_3			
$\Rightarrow$	24 ;	12 ;	6			
=	4×6 ;	2×6 ;	1×6 ;	$\frac{1}{2}\times 6$;	$\frac{1}{4}\times 6$;	$\frac{1}{8}\times 6$
=	$2^2\times 6$;	$2^1\times 6$;	$2^0\times 6$;	$2^{-1}\times 6$;	$2^{-2}\times 6$;	$2^{-3}\times 6$

$= \quad 2^3 \times 6\ ; \quad 2^2 \times 3\ ; \quad 2^1 \times 3\ ; \quad 2^0 \times 3\ ; \quad 2^{-1} \times 6; \quad 2^{-2} \times 3$

Next 3 terms will be: $3; \frac{3}{2}; \frac{3}{4}$

$T_n = 2^{4-n}.3$

Alternatively

24; 12; 6; …

$24; \frac{24}{2}, \frac{24}{4}, \frac{24}{8}, \frac{14}{16}, \frac{24}{32}, \ldots$

$Tn = 24.\frac{1}{2}^{n-1}$

Both these general terms are the same:

$T_n = 2^{4-n}.3$

and $T_n = 24.\left(\frac{1}{2}\right)^{n-1}$

$= 8.3.2^{1-n}$

$= 2^3.3.2^{1-n}$

$= 3.2^{4-n}$

Example 3 (Problem solving)

A rubber ball is dropped from a height of 30m. After each bounce, it returns to a height that is 4/5 of the previous height.

a) express the first three heights as a sequence

Solution

$30\left(\frac{4}{5}\right); 30\left(\frac{4}{5}\right)^2, 30\left(\frac{4}{5}\right)^3$

b) how high will the ball be after 21 bounces?

Solution

$T_{21} = 30\left(\frac{4}{5}\right)^{21}$

Activity 1

FORMATIVE ASSESSMENT

PAIRS

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Example 4 (The Fibonacci sequence)

Write down the next three terms in the sequence 1; 1; 2; 3; 5; 8;

Solution

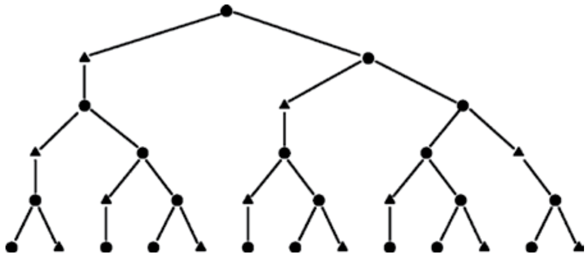
13; 21; 34

These are fascinating numbers because they appear all over in our world.

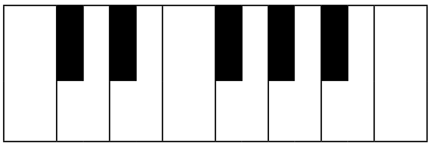
THE FAMILY

(In this family tree a male is represented by the symbol (▲) and a female by the symbol ●)

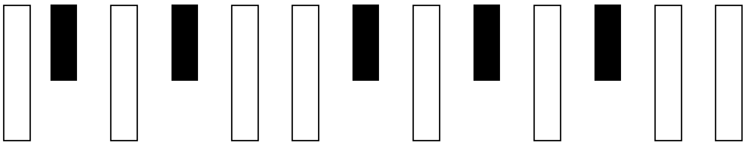
Generation back	Number of bees in each generation
1st	1
2nd	2
3rd	3
4th	5
5th	8
6th	13



The 13 keys shown below form one octave of a piano.



If we spread them out



What do we notice?

5 black keys

8 white keys

13 keys

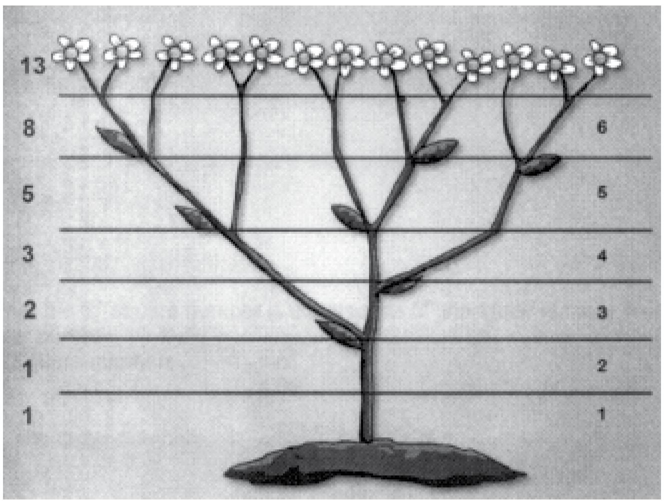
ALL Fibonacci numbers

In the 6th generation back of the male bee we have

● ▲ ● ▲ ● ● ▲ ● ▲ ● ▲ ● ●

Males are the black keys and females the white keys.

Look at this example from the internet.



www.mcs.surrey.ac.uk/personal/R.Knott/Fibonacci/Fibonat.html

The golden ratio

1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; 144

$\frac{T2}{T1}$= 1; $\frac{T3}{T2T}$=T2; $\frac{T4}{T3T}$=T1,5; $\frac{T5}{T4}$= 1,666; $\frac{T6}{T5T}$=T1,6

$\frac{13}{8}$= 1,625; $\frac{21}{13}$= 1,61538…; $\frac{34}{21}$= 1,61904; $\frac{55}{34}$= 1,6176…

$\frac{89}{55}$= 1,618618…; $\frac{144}{89}$= 1,617977…

Notice, as the sequence gets larger so the ratio gets closer to 1,62.

We call this the GOLDEN RATIO.

Workbook: Lesson 21

Activity 2

Triangular numbers

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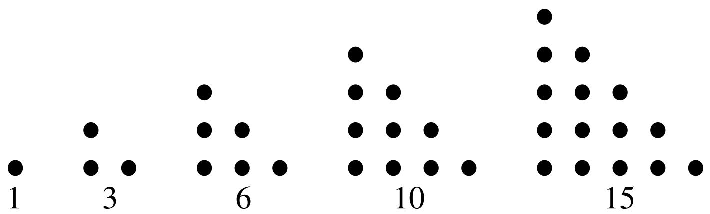
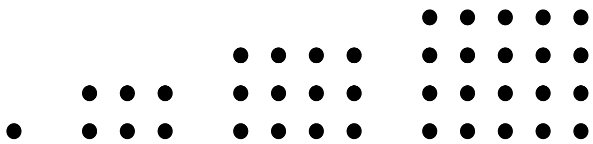
Numbers that form triangles



Look at the picture and find the rule.

Lets make rectangles



$T_1 = 1$

$T_2 = 2 \times \frac{3}{2}$

$T_3 = 3 \times \frac{4}{2}$

$T_4 = 4 \times \frac{5}{2}$

$T_n = \frac{n(n+1)}{2}$

Show the 6th square number is equal to the 5th triangular number plus the 6th triangular number.

Square numbers: $T_n = n^2$

$T_6 = 36$

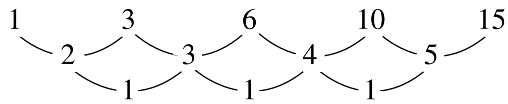
Triangular numbers $T_n = \frac{n(n+1)}{2}$

$T_5 = 15$ $T_6 = 21$

$21 + 15 = 36$

Try more examples

Alternate exploration:



	T_1	T_2	T_3	T_4
$\frac{1}{2}n^2$	$\frac{1}{2}$	2	$\frac{9}{2}$	8
	1	3	6	10
snt	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$

Make a conjecture

The nth square is equal to the some of the nth and the $(n-1)^{th}$ triangular numbers.

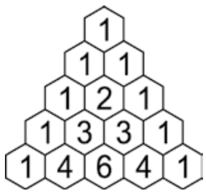
Now prove it: Formula for the triangular numbers

$\frac{n(n+1)}{2}$ so $\frac{n(n+1)}{2} + \frac{(n-1)(n)}{2} = \frac{n^2+n+n^2-n}{2}$

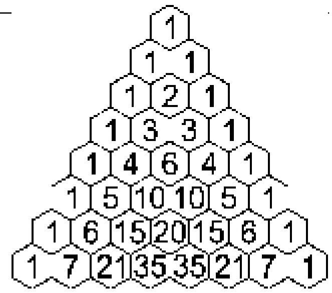
$= \frac{2n^2}{2}$

$= n^2$

PASCAL'S TRIANGLE

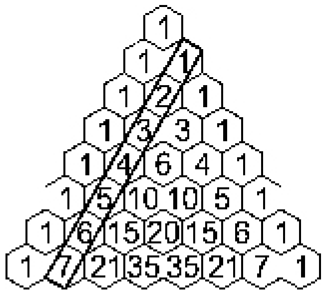


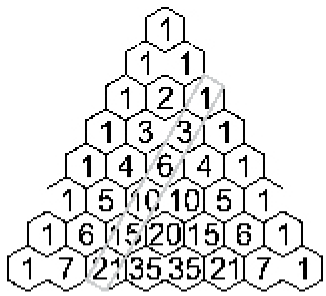
Complete the next three lines



a) Look for a linear sequence

b) Look for triangular numbers





c) Find the sum of the terms in the 51^{st} row

Row	Sum	Gen
1	1	20
2	2	21
3	4	22
4	8	23
5	16	24
6	32	25

1^{st} Row 1 2^{nd} Row 2 3^{rd} Row 4 4th Row 8

5th Row 2^4 6th Row 2^5 nth Row 2^{n-1} 51^{st} Row = 2^{50}

So $T_n = 2^{n-1}$

$T_{51} = 2^{50}$

WORKBOOK: Lesson21

Activity 1

1. Find the nth term of the following sequences.

   a) $4; -2, 1; -\frac{1}{2}, \frac{1}{4}, \ldots$

   b) $-\frac{1}{8}, -\frac{1}{2}, -2; -8; \ldots$

   c) $32; 16; 8; 4; 2; \ldots$

   d) $3a; 6a^2; 9a^3; 18a^4 \ldots$

2. a) Find T_n

   b) Find the 8th term 9; 3;1 …

3. Which term of the sequence

   $1; \frac{3}{2}, \frac{9}{4} \ldots$ is equal to $\frac{243}{32}$? (Hint; Find T_n first)

4. A tree grows 120 cm during the first year. Each year it grows $\frac{9}{10}$ of the previous year's growth. How much in the 58th year?

Activity 2

Write a paragraph on Leonardo Fibonacci.

Find out all you can about the Fibonacci numbers and where we find them in nature (you may look at various flowers, spirals, shells, the pineapple).

Generate the Fibonacci sequence by looking at the reproduction of rabbits.

Investigate the Golden Ratio and why it represents a "mathematically ideal" ratio.

Investigate the history of the golden ratio, including its role in the ancient Greek architecture.

Learn how to construct the Golden Rectangle.

You must supply a reference to the sources you used including web addresses.

Activity 3

1. a) Show why the 5th triangular number is $T_5 = \frac{1}{2} \times 5 \times 6$

   b) Find an expression in n for the nth triangular number.

2. a) Use the diagrams of the 3rd and 4th triangular numbers, T_3 and T_4 to show that their sum that their sum is the 4th square number, i.e. $S_4 = T_3 + T_4$

   b) Generalise with a formula the connection between triangular and square numbers.

3. a) Write down a table of square numbers from the 1st to the tenth.

   b) Find two square numbers which add to give a square number.

   c) Repeat part (b) for at least three other pairs of square numbers.

4. Without using a calculator explain whether:

a) 441 b) 2001

c) 1007 d) 4096 is a square number

5. Show that the difference between any two consecutive square numbers is an odd number.

6. Show that the difference between

a) The 7th square number and the 4th square number is a multiple of 3.

b) The 8th square number and the 5th square number is a multiple of 3.

c) The 11th square number and the 7th square number is a multiple of 4.

d) Generalise the statement implied in parts (a) (b) and (c).

e) 64 is equal to the 8th square number $S_8 = 8^2$

64 is equal to the 4th cube number $C_4 = 4^3$

Find other cube numbers which are also square numbers.

If you can, make a general comment about such cube numbers.

7. For any positive whole number n its "Tan function" $t(n)$ is defined as the number of positive whole number factors of n.

7 is a prime number. It has two factors so $t(7) = 2$

a) Show that if p is any prime number the $t(p) = 2$

b) For any prime number p and any positive value of n, find an expression for $t(p^n)$.

c) $6 \times 7 = 42$

The factors of 42 are:

1, 2, 3, 6, 7, 14, 21, 42 so $t(42) = 8$ and $t(6) = 4$

$t(7) = 2$

$\therefore\ t(6) \times t(7)\ n = 8$ so $t(6 \times 7) = t(6) \times t(7)$

Investigate to see whether $t(n \times m) = t(n) \times t(m)$

8. One way of making the number 5 by adding ones and threes is:

5 = 3 + 1 + 1 … and another different way is: 5 = 1 + 3 + 1

Investigate the number of different ways of making any number by adding ones and threes.

9.

$1^3 + 2^3$	$1^3 + 2^3 + 3^3 + 4^3$	$1^3 + 2^3 + 3^3$
$= 1 + 8$	$= 1 + 8 + 27 + 64$	$= 1 + 8 + 27$
$= 9$	$= 100$	$= 36$
$= 3^2$	$= 10^2$	$= 6^2$
$= (1 + 2)^3$	$= (1 + 2 + 3 + 4)^2$	$= (1 + 2 + 3)^2$

Investigate this situation further. Try other powers.

10. The last digit of 146 is 6. this is written LD(146) = 6. What comments can you make about

a) LD($n \times m$)

b) LD(any square number)

c) Show that $10n + 7$ can never be a square number for any positive whole number value of n.

11. Mystery sequences

Find the next three terms of the following sequences:

6; 8; 12; 14; 18 …

767; 294; 72; …

1; 5; 12; 22 … and find T_n