ARITHMETIC SEQUENCES

Lesson 1

Learning Outcomes and Assessment Standards

Learning outcome 1: Number and number relationships
Assessment standard 12.1.3

- Identify and solve problems involving number patterns, including but not limited to arithmetic and geometric sequences and series.
- Correctly interpret sigma notation.
- Prove and correctly select the formula for and calculate the sum of series, including:

$$\sum_{i=1}^{n} 1 = n; \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}; \quad \sum_{i=1}^{n} a + (i-1)d = \frac{n}{2}[2a + (n-1)d]$$

$$\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n - 1)}{r-1};\ r \neq 1; \quad \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r};\ -1 < r < 1$$

- Correctly interpret recursive formulae: (e.g. $T_{n+1} = T_n + T_{n-1}$)

Overview

In this lesson you will:

- identify an arithmetic sequence
- derive a formula to find the n^{th} term of an arithmetic sequence
- work through problems involving arithmetic sequences.

Lesson

LC DVD

Look at these sequences and supply the next three terms and in each case a formula to find the n^{th} term (T_n)

1. 2 ; 5 ; 8 ; 11 ; 14

   +3 +3 +3 +3 → First order difference is constant. Thus the relationship is linear.

   So $T_1 = 2$
   $T_2 = 2 + 3$
   $T_3 = 2 + 3 + 3 = 2 + 2(3)$
   $T_4 = 2 + 3 + 3 + 3 = 2 + 3(3)$
   $T_5 = 2 + 4(3)$

   } Each term can be expressed in terms of the first term and the constant difference.

   $T_n = 2 + (n-1)(3) = 2 - 3 + 3n = 3n - 1$

2. 8 ; 3 ; –2 ; –7 ; –12

   –5 –5 –5 –5 → First order difference is constant. Thus the relationship is linear.

   So $T_1 = 8$
   $T_2 = 8 + 1(5)$
   $T_3 = 8 + 2(-5)$
   $T_4 = 8 + 3(-5)$
   $T_5 = 8 + 4(-5)$

   } Each term can be expressed in terms of the first term and the constant difference.

   $T_n = 8 + (n-1)(-5) = 8 + 5 - 5n = 13 - 5n$

These sequences are called arithmetic sequences because the first order difference is a constant.

Example

$(3x - 1)$; $(3x + 3)$; $(4x + 3)$ form an arithmetic sequence, find the first three terms and a rule for T_n.

Solution

Since the sequence is arithmetic, the differences between terms are constant

$$T_2 - T_1 = T_3 - T_2$$
$$(3x + 3) - (3x - 1) = (4x + 3) - (3x + 3)$$
$$3x + 3 - 3x + 1 = 4x + 3 - 3x - 3$$
$$4 = x$$

$$11 \; ; \; 15 \; ; \; 19 \; ; \; 23 \; ; \; \ldots$$
$$+4 \quad +4 \quad +4$$

$$T_n = 11 + (n - 1)(4) = 11 - 4 + 4n = 4n + 7$$

General rules

Let the first term be a

$T_1 = a$
$T_2 = a + d$
$T_3 = a + 2d$
$T_4 = a + 3d$
$T_{20} = a + 19d$
$T_n = a + (n - 1)d$

- a is the first term
- d is the common difference
- T_n is the value of the term
- n is the term number

Examples

1. Find the 26th term of the sequence 2; 5; 8; 11

Solution

$$T_{26} = a + 25d$$
$$a = 2 \quad d = 3 \quad T_{26} = ?$$
$$T_{26} = 2 + 25(3) = 77$$

2. Which term in the sequence 10; 6; 2; –2; … will have a value of –58?

Solution

$$n = ? \quad a = 10 \quad d = -4 \quad T_n = -58$$
$$T_n = a + (n - 1)d$$
$$-58 = 10 + (n - 1)(-4)$$
$$-68 = -4n + 4$$
$$4n = 72$$
$$n = 18$$

3. Find the arithmetic sequence with a 4th term of 3 and an 8th term of –13.

Solution

$a + 3d = 3$

$a + 7d = -13$

Solve simultaneously to find a and d.

$a + 7d = -13$

$\underline{a + 3d = 3}$ Subtract to eliminate a

$4d = -16$

$d = -4$

Substitute to get a

$a - 12 = 3$

$a = 15$

Sequence 15; 11; 7; …

4. Tsolo is training for the Comrades Marathon. For the first week he runs 4 km per day. With each consecutive week he increases his daily distance by 3 km for the week. That will mean that he runs 7 km per day for the next week, and so on. He will be ready for the Comrades if he can run a distance of 94 km in one day. How many days will he have to train?

Solution

Week 1 → 4 km per day for 7 days

Week 2 → 7 km per day for 7 days

Week 3 → 10 km per day for 7 days

So 4 ; 7 ; 10 ; 13 ; 16 ; … ; 94

3 3 3 3

$T_n = a + (n - 1)d$

$94 = 4 + (n - 1)(3)$

$90 = 3(n - 1)$

$n - 1 = 30$

$n = 31$

He will be able to run 94 km in the 31st week from now, so he should train for a minimum of 30 weeks.

Activity

1. The 10th term of an A.P. is 17 and the 16th term is 44. Find the first three terms in the sequence.
2. The n^{th} term of the sequence 4; 11; 18; ……. is 697. Find n.
3. Given the sequence $T_q = -7 + 3(q + 4)$ Find:
   a) T_4
   b) x if $T_x = 50$
4. The first three terms in an AP is given by x; $4x + 5$ and $10x - 5$. Find the value of x.

5. The first three terms of an AP are $x - 1$; t and $3x+1$.

   a) Show that $t = 2x$

   b) Find the common difference in terms of x

   c) Find x if it is given that $T_{10} = 108$

6. The seventh, eight and thirteenth terms of an AP are $3x - 1$; x and $20 - 8x$. Find the value of x.

7. Insert 5 terms between 7 and –11 so that the seven terms form an AP. (these five terms are called arithmetic means)

8. Which is the first term in the sequence 5; 13; 21; … to exceed 1 569?

9. The sum of the first two terms of an AP is 22 and the third term is 8. Find the 10^{th} term.

10. In order to complete the training schedule for a marathon an athlete needs to reach a target of 94 kms a week. Rory increases his distance by $3\frac{1}{2}$ km per week. If he starts at 8 km, how long will it take him to reach this target?

11. Michael can do 24 sit ups per minute. Each week he improves his record by 4 sit ups per minute. Robbie can do 20 sit ups per minute and he increases his record by 5 sit ups per minute each week.

    a) After how many weeks will their number of sit ups per minute be the same?

    b) Who will win the fitness competition in six weeks time?

12. The price of each casserole dish in a set of 5 dishes decreases by equal amounts. The smallest and the middle sized casserole together cost R20.

    The 2^{nd} largest costs twice as much as the smallest.

    Calculate the cost of each dish in the set.