Lesson 6

SEQUENCES AND SERIES

Recursive relationships and expressions

Learning Outcomes and Assessment Standards

Learning outcome 1: Number and number relationships
Assessment standard 12.1.3

- Identify and solve problems involving number patterns, including but not limited to arithmetic and geometric sequences and series.
- Correctly interpret recursive formulae: (e.g. $T_{n+1} = T_n + T_{n-1}$)

Overview

In this lesson you will:

- generate sequences by using a recursive rule
- formulate a recursive rule.

Lesson

DVD

9 11 13 15 17 …
+2 +2 +2 +2

This is an arithmetic sequence since the first order difference is a constant positive 2.

The general rule for this sequence will thus be $T_n = 2n + 3$.

To express the relationship as a recursive one, we need to consider what happens from one term to the next. That will mean that we have to express the terms as a relationship with it's previous term. It is clear to see that the next term is the previous term plus two. So we write:

$$T_n = T_{n-1} + 2$$

This establishes the relationship between the terms, but it is important to know the first term so that we can generate the sequence. So the complete recursive relationship will thus be expressed as:

$$\begin{cases} T_n = T_{n-1} + 2 \\ T_1 = 5;\ n \in \mathbb{N} \end{cases} \text{ or we can say } \begin{cases} T_{n+1} = T_n + 2 \\ T_1 = 5;\ n \in \mathbb{N} \end{cases}$$

Remember, if we talk about consecutive terms we say:

… $n - 2$; $n - 1$; n; $n + 1$; $n + 2$ …

So if we say T_n, the previous term will be T_{n-1}.

If the sequence started at 9:

9 11 13 15 17 …
+2 +2 +2 +2

The general rule will now be: $T_n = 2n + 3 + 4 = 2n + 7$. We saw this because if $n = 1$, then $T_1 = 2(1) + 3 = 5$ and we need to add 4 to get to 9.

The recursive relationship stays the same, but it just starts later. So it will be:

$$\begin{cases} T_n = T_{n-1} + 2 \\ T_1 = 9;\ n \in \mathbb{N} \end{cases} \text{ or we can say } \begin{cases} T_{n+1} = T_n + 2 \\ T_1 = 9;\ n \in \mathbb{N} \end{cases}$$

The first order difference is not constant:

Consider the sequence: –2 ; 0 ; 6 ; 16 ; 30 ; …

-2 0 6 16 30 ....

+2 +6 +10 +14 ⟶ First order difference is not constant

4 4 4 ⟶ Second order difference is constant

So for the general rule:

T_1	T_2	T_3	T_4	T_5	
2	8	18	32	50	$T_n = 2n^2 + \ldots$
–2	0	6	16	30	What I want
–4	–8	–12	–16	–20	What I am short

Focusing on: –4 ; –8 ; –12 ; –16 ; –20 ; … $= -4 + (n-1)(-4) = -4n$

So $T_n = 2n^2 - 4n;\ n \in \mathbb{N}$

$T_1 = -2$

$T_2 = T_1 + 2$

$T_3 = T_2 + 6$

$T_4 = T_3 + 10$

$T_5 = T_4 + 14$

To express the same relationship recursively:

–2

+2 +6 +10 +14

$T_1 = -2$

$T_2 = T_1 + 2$

$T_3 = T_2 + 6$

$T_4 = T_3 + 10$

$T_5 = T_4 + 14$

We focus on: 2 ; 6 ; 10 ; 14 ; … $= 2 + (n-1)(4) = 4n - 2$.

(Since the relationship is arithmetic)

So: $\begin{cases} T_{n+1} = T_n + 4n - 2 \\ T_1 = -2;\ n \in \mathbb{N} \end{cases}$

Be very careful here:

Notice that we expressed the relationship as $\begin{cases} T_{n+1} = T_n + 4n - 2 \\ T_1 = -2;\ n \in \mathbb{N} \end{cases}$ starting with T_{n+1}.

To check the answer we can say let $n = 1$ in $T_{n+1} = T_n + 4n - 2$:

$T_{1+1} = T_1 + 4(1) - 2$

$\therefore\ T_2 = T_1 + 2$

This is true if we look at the sequence.

If we chose to express this relationship starting with $T_n = T_{n-1} + \ldots$, then $n = 2$ will be our first substitution. So the rule becomes $T_n = T_{n-1} + 4(n-1) - 2 = T_{n-1} + 4n - 6$. So we can write: $\begin{cases} T_n = T_{n-1} + 4n - 6 \\ T_1 = -2;\ n \in \mathbb{N} \end{cases}$

So the two recursive rules are: $\begin{cases} T_{n+1} = T_n + 4n - 2 \\ T_1 = -2;\ n \in \mathbb{N} \end{cases}$ or $\begin{cases} T_n = T_{n-1} + 4n - 6 \\ T_1 = -2;\ n \in \mathbb{N} \end{cases}$

More examples

1. Give the first four terms of the following sequences:

   a) $T_1 = 6$ and $T_n = T_{n-1} + 4$

   Solution 6; 10; 14; 18

   b) $T_1 = \frac{1}{2}$ and $T_n = T_{n-1} \times 2$

   Solution $\frac{1}{2}$; 1; 2; 4

   c) $T_1 = 2$ and $T_n = (T_{n-1})^2$

   Solution 2; 4; 16 ; 256

2. Give a recursive formula for each of the following sequences.

   a) 3; 6; 9; 12; …

   Solution $T_n = T_{n-1} + 3$
   $T_1 = 3$

   b) (–2)10; –20; 40; –80

   Solution $T_n = T_{n-1} \times$
   $T_1 = 10$

   c) 1; 1; 2; 3; 5; 8; …

   Solution $T_n = T_{n-1} + T_{n-2}$
   $T_1 = 1 \quad T_2 = 1$

3. a) An estimated 40,3 million people were living with HIV in 2005.

   In 2003 an estimated 37,5 million people had HIV. Assuming that the percentage increase was the same each year, what was the percentage increase each year from 2003 to 2005?

   b) Write a recursive rule for this sequence as $T_n = T_{n-1}$ …

   c) If the number of people infected with HIV continues growing at the same rate, in what year will there be more than 50 million people who are HIV positive?

Solution

a)
$$T_1 = 37{,}5 \text{ million}$$
$$a = 37{,}5 \text{ million}$$
$$T_3 = 40{,}3 \text{ million}$$
$$ar^2 = 40{,}3 \text{ million}$$
$$37{,}5r^2 = 40{,}3$$
$$r^2 = \frac{40{,}3}{37{,}5}$$
$$r = 1{,}03661308$$

2003: 37,5 million 2004: 38,87 million 2005: 40,3 million

Percentage increase: 3,66%

b) $T_n = T_{n-1} \times (1{,}0366......)$

$T_1 = 37{,}5$

c) Use the geometric sequence formula

$$ar^{n-1} = 50$$

$$(1{,}0366)^{n-1} = \frac{50}{37{,}5}$$

$$\log(1{,}0366)^{n-1} = \log\left(\frac{50}{37{,}5}\right)$$

$$n - 1 = \frac{\log\left(\frac{50}{37{,}5}\right)}{\log 1{,}0366}$$

$$(37{,}5)(1{,}0366)^{n-1} = 50$$

$n = 9$ Year 2012

4. A new house costs R440 000. It is estimated that the price of residential properties in that town increases at a rate of 12% per annum compounded annually.

   a) Write down the price of the new house as a sequence for the next four years.

   b) What will the same house cost in 10 years time?

   c) Write down a recursive formula.

   Solution

   a) 440 000; 440 000(1,12); 440 000$(1{,}12)^2$; 440 000$(1{,}12)^3$

   b) $ar^{n-1} = 440\,000(1{,}12)^9$

   $=1\,220\,154{,}65$

   c) $T_n = T_{n-1} \times (1{,}12)$

Activity 1

Consider the sequences below and write down a general rule for each sequence, as well as a recursive relationship for each:

1. 5 ; 8 ; 13 ; 20 ; 29 ; 40; …
2. 1 ; 2 ; 9 ; 22 ; 41 ; 66; …
3. –2 ; 2 ; 9 ; 19 ; 32 ; 48; …

ANSWERS AND ASSESSMENT

Lesson 1

1. $a + 9d = 17$ … (1) and $a + 15d = 44$ … (2).
   So (2) – (1) : $6d = 27 \therefore d = 9$
   Then $a = 17 - 9\left(\frac{9}{2}\right) = -47$
   The sequence: $-\frac{47}{2}; -19; -\frac{29}{2}; \ldots$
2. $T_n = a + (n-1)d \Rightarrow 697 = 4 + (n-1)7$
   $\therefore n - 1 = 99 \Rightarrow n = 100$
3. a) $T_q = -7 + 3(q+4) \Rightarrow T_4 = -7 + 3(8) = 17$
   b) $Tx = -7 + 3(x+4) = 50 \Rightarrow (x+4) = 19$
   $\therefore x = 15$
4. $T_2 - T_1 = T_3 - T_2 \Rightarrow 4x + 5 - x = 10x - 5 - 4x - 5$
   $\therefore -3x = -15$
   $\therefore x = 5$
5. a) $T_2 - T_1 = T_3 - T_2 \Rightarrow t - x + 1 = 3x + 1 - t$
   $\therefore 2t = 4x \Rightarrow t = 2x$
   b) $T_2 - T_1 = 2x - x + 1 = x + 1$
   c) $T_n = a(n-1)d \Rightarrow 108 = x - 1 + (10-1)(x+1)$
   $\therefore 108 = 10x + 8 \Rightarrow 100 = 10x \Rightarrow x = 10$
6. $a + 6d = 3x - 1$ … (1)
   $a + 7d = x$ … (2)
   $a + 12d = 20 - 8x$ … (3)
   From (2) – (1) : $d = -2x + 1$ … (4)
   From (3) – (2) : $5d = 20 - 9x$ … (5)
   Take (4) × –5 : $-5d = 10x - 5$ and add this to (5) to get $0 = x + 15$.
   So $x = -15$.
7. $T_1 = a = 7$ and $T_7 = a + 6d = -11$.
   So if $a = 7 : 7 + 6d = -11 \rightarrow 6d = -18 \rightarrow d = -3$.
   So the terms are: 7; 7 – 3; 7 – 2(3); 7 – 3(3); 7 – 4 (3); 7 – 5(3); – 11
   $\therefore$ 7; 4; 1; – 2; – 5; – 8; – 11
8. $T_n = a + (n-1)d \Rightarrow 5 + (n-1)8 > 1\,569$
   $\therefore 8(n-1) > 1\,564$
   $\therefore (n-1) > \frac{391}{2}$
   $\therefore n > \frac{393}{2}$
   $\therefore n = 197$
9. $a + (a + d) = 22$ and $a + 2d = 8$.
   So $2a + d = 22$ … (1) and $a + 2d = 8$ … (2)
   So if we take (1) × –2 : $-4a - 2d = -44$ and we add it to (2) we get $-3a = -36$
   $\therefore a = 12$
   Then : $12 + 2d = 8 \rightarrow 2d = -4 \rightarrow d = -2$
   So $T_{10} = a + 9d = 12 - 18 = -6$.

Lesson 2

1. The sequence.
   $p + 3p + 5p + - - - - -$
   $a = p \quad d = 2p \quad n = p$
   $Sp = \frac{p}{2}[2p + (p-1)2p]$
   $= \frac{p}{2}[2p + 2p^2 - 2p]$
   $= \frac{p}{2}[2p^2] = p^3$
   OR
   $\sum_{i=1}^{p}(2i-1)p = p^3$
   so $\sum_{i=1}^{p}(2i-1)p = \sum_{i=1}^{p}(2ip - p)$
   $= 2p\left(\sum_{i=1}^{p} i\right) - p\sum_{i=1}^{p} 1$
   $= 2p\left[\frac{p}{2}(p+1)\right] - p.p$
   $= p^2(p+1) - p^2$
   $= p^3 + p^2 - p^2$
   $= p^3$
2. $S_{20} = 1580$
   Using $S_n = \frac{n}{2}[2a + (n-1)d]$:
   $1\,580 = \frac{20}{2}[2a + 19d]$
   $158 = 2a + 19d$ …(A)
   $S_{25} - S_{20} = 895$
   $\frac{25}{2}[2a + 24d] - 1\,580 = 895$
   $\therefore 25a + 25(12)d = 2\,475$
   $a + 12d = 99$ … (B)
   Now: (A) – 2(B):
   $158 = 2a + 19d$
   $-198 = -2a - 24d$
   Add $-40 = -5d$
   $d = 8$
   Then $a = 99 - 12(8) = 3$
3. $\sum_{r=1}^{n}\frac{1}{3}(25 - r) = 95$
   $\frac{25}{3}\left(\sum_{r=1}^{n} 1\right) - \frac{1}{3}\left(\sum_{r=1}^{n} r\right) = 95$
   $\therefore \frac{25}{3}(n) - \frac{1}{3}\left(\frac{n}{2}(n+1)\right) = 95$
   $\therefore 25n - \frac{n}{2}(n+1) = 285$
   $\therefore 50n - n^2 - n = 570$
   $\therefore n^2 - 49n + 570 = 0$
   $T_1 = \frac{1}{3}(25 - 1) = 8$
   $T_n = \frac{1}{3}(25 - n)$
   $\therefore 95 = \frac{n}{2}\left[8 + \frac{1}{3}(25 - n)\right]$
   $\therefore 190 = 8n + \frac{n}{3}(25 - n)$
   $\therefore 570 = 24n + 25n - n^2$
   $\therefore n^2 - 49n + 570 = 0$
   Now: $(n - 30)(n - 19) = 0$
   $\therefore n = 30$ or $n = 19$

4. $T_1 = (1 - x)^2 = a$

$d = T_2 - T_1$

$= 1 + x^2 - 1 + 2x - x^2$

$= 2x$

$S_{10} = 5\,[2a + 9d] = 310$

$\Rightarrow 2a + 9d = 62$

$2(1 - 2x + x^2) + 18x = 62$

$1 - 2x + x^2 + 9x = 31$

$x^2 + 7x - 30 = 0$

$(x + 10)(x - 3) = 0$

$x = -10$ or $x = 3$

5. a) $\frac{1}{101} + \frac{2}{101} + \frac{3}{101} + \ldots + \frac{100}{101}$

Method 9:

$= \frac{\frac{100}{2}[101]}{101}$ Using $\sum_{i=1}^{n} i = \frac{n}{2}(n+1)$

$= 50$

Method 2:

$\frac{1}{101}[1 + 2 + 3 + \ldots + 100]$

$a = 1$ $d = 1$ $n = 100$

$S_{100} = 50[1 + 100]$

(Using $S_n = \frac{n}{2}[a + T_n]$

$= 50(101)$

So $\frac{1}{101} \times 50(101) = 50$

b) $\frac{1}{2} + 1 + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right)$

$+ \left(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right)$

$+ \left(\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6}\right)$

$+ \ldots + \left(\frac{1}{101} + \ldots + \frac{100}{101}\right)$

$= \frac{1}{2} + 1 + \frac{1}{4}\left(\frac{3}{2}\right)(4) + \frac{1}{5}\left(\frac{4}{2}(5)\right) + \frac{1}{6}\left(\frac{5}{2}(6)\right)$

$+ \ldots + \frac{1}{101}(50)(101)$

$= \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + \ldots + 50$

$a = \frac{1}{2}$ $d = \frac{1}{2}$ $a + (n - 1)d = 50$

$T_n = 50$ $\frac{1}{2} + (n - 1)\frac{1}{2} = 50$

$1 + n - 1 = 100$

$\therefore n = 100$

So $S_{100} = \frac{100}{2}\left(\frac{1}{2} + 50\right)$

$= 50\left(\frac{1}{2} + 50\right)$

$= 2\,525$

6. $\sum_{k=n+1}^{2n} k = (n + 1) + (n + 2) + (n + 3) + \ldots + 2n$

$a = n + 1$ For the number of terms:

$d = 1$ (Top – Bottom) + 1

$T_n = 2n$ $= 2n - (n + 1) + 1$

$= 2n - n - 1 + 1$

$= n$

$\therefore S_n = \frac{n}{2}[a + T_n]$

$= \frac{n}{2}[n + 1 + 2n]$

$S_n = \frac{n}{2}[3n + 1]$

7. $T_n = x$ $a + (n - 1)d = x$

$T_{n+1} = a + (n + 1 - 1)d = y$

$a + nd = y$

$a + n(y - x) = y$

$a + ny - nx = y$

$a = y + nx - ny$

8. $a = 6$ $d = -5$ $T_n = -239$

$a + (n - 1)\,d = -239$

$6 - 5n + 5 = -239$

$-5n = -250$ $\therefore n = 5a$

$S_n = \frac{n}{2}(a + \ell)$

$S_{50} = 25(6 - 239)$

$= 25\,(-233) = -5825$

9. $\sum_{r=1}^{n}(95 - 7r) = 100$

$\therefore 95\sum_{r=1}^{n} 1 - 7\left(\sum_{r=1}^{n} r\right) = 100$

$\therefore 95n - 7\frac{n}{2}(n + 1) = 100$

$\therefore 190n - 7n^2 - 7n = 200$

$\therefore 7n^2 - 183n + 200 = 0$

$T_1 = 95 - 7 = 88$

$T_n = 95 - 7n$

$\therefore S_n = \frac{n}{2}[88 + 95 - 7n]$

$\therefore 100 = \frac{n}{2}[183 - 7n]$

$\therefore 200 = 183n - 7n^2$

$\therefore 7n^2 - 183n + 200 = 0$

So $(7n - 8)(n - 25) = 0$

$n = 25$ (since $n \in \mathbb{N}$)

10. a) $S_8 = 8^2 - 2(8)$

$= 64 - 16 = 48$

b) $T_8 = S_8 - S_7$

$S_7 = 7^2 - 2(7)$

$= 49 - 14 = 35$

$\therefore T_8 = 48 - 35 = 13$

Lesson 3

1. $\frac{T_2}{T_1} = \frac{T_3}{T_2} \rightarrow (T_2)^2 = T_1 \cdot T_3$

$\therefore \left(\frac{1}{6}\right)^2 = -\frac{1}{4} \times p$

$\therefore \frac{1}{36} = -\frac{p}{4}$

$\therefore p = -\frac{1}{9}$

2. *a)* $\frac{T_2}{T_1} = \frac{T_3}{T_2} \rightarrow (T_2)^2 = T_1 \cdot T_3$

$\therefore (t - 3)^2 = (t - 1)(t - 4)$

$\therefore t\,2 - 6t + 9 = t\,2 - 5t + 4$

$\therefore -t = -5$

$\therefore t = 5$

b) $T_{20} = ar^{19}$ if the terms are 1 ; 2 ; 4

$\therefore T_{20} = 1 \times 2^{19}$

3. $a = 0{,}06 = 6/100 = 3/50$

   $r = -0{,}18/0{,}06 = -3$

   $T_n = a \cdot rn - 1 = 43{,}74$

   $\therefore (3/50) \cdot (-3)n - 1 = 43{,}74$

   $\therefore (-3)^{n-1} = 729 = 3^6$

   $\therefore n - 1 = 6$

   $\therefore n = 7.$

4. $\Rightarrow T_7 = ar^6$ and $T_4 = ar^3$

   $ar^7 = \frac{4}{243}$ and $ar^3 = \frac{4}{9}$

   To find the sequence, we need to find a and r.

   $\frac{T_7}{T_4} = \frac{ar^6}{ar^3} = \frac{\frac{4}{243}}{\frac{4}{9}} \to r^3 = \frac{9}{243} = \frac{1}{27}$

   $\therefore r = \frac{1}{3}$

   Then: $ar^3 = \frac{4}{9}$

   $a\left(\frac{1}{3}\right)^3 = \frac{4}{9} \to \frac{a}{27} = \frac{4}{9}$

   $\therefore 9a = 108 \to a = 12$

   Thus: $a; ar; ar^2; \ldots = 12; 4; \frac{4}{3};$

5. $\Rightarrow$ It is given that the sequence is a G.P., so we use r to find the value of p

   $\frac{T_5}{T_4} = \frac{T_6}{T_5}$

   $\therefore \frac{p+2}{p-4} = \frac{3p+1}{p+2}$

   $\therefore (p+2)^2 = (3p+1)(p-4)$

   $\therefore p^2 + 4p + 4 = 3p^2 - 11p - 4$

   $\therefore 2p^2 - 15p - 8 = 0$

   $\therefore (2p+1)(p-8) = 0$

   $\therefore p = -\frac{1}{2}$ OR $p = 8$

   Thus for $p = -\frac{1}{2}$: T_4 ; T_5 ; T_6

   $= -\frac{1}{2} - 4; -\frac{1}{2} + 2; 3\left(-\frac{1}{2}\right) + 1$

   $= -\frac{9}{2}; \frac{3}{2}; -\frac{1}{2}$

   so $r = -\frac{\frac{3}{2}}{\frac{9}{2}} = -\frac{3}{9} = -\frac{1}{3}$

   and for $p = 8$: T_4; T_5; T_6 $= 8 - 4; 8 + 2; 3(8) + 1$

   $= 4; 10; 25$

   so $r = \frac{10}{4} = \frac{5}{2}$

6. $\frac{x}{15} = \frac{60}{x}$

   $\therefore x = \pm\sqrt{15 \cdot 60}$

   $\therefore x = \pm\sqrt{900}$

   $\therefore x = \pm 30$

   Thus the sequences can be : 15; 30; 60 or 15 ; −30 ; 60

7. $\Rightarrow$ The sequence is 3; x; y; z; 243

   thus: $a = 3$ and $T_5 = 243$

   $\therefore a = 3$ and $ar^4 = 243$

   $\therefore 3r^4 = 243$

   $\therefore r^4 = \frac{243}{3}$

   $\therefore r^4 = 81$

   $\therefore r = \pm 3$

   now : if $r = 3$ and $a = 3$ or if $r = -3$ and $a = 3$

   $\therefore x = ar = 3 \cdot 3 = 9$ $\therefore x = ar = -9$

   $\therefore y = xr = 9 \cdot 3 = 27$ $\therefore y = xr = 27$

   $\therefore z = yr = 27 \cdot 3 = 81$ $\therefore z = yr = -81$

   The sequence : 3 ; 9 ; 27 ; 81 ; 243 or 3 ; −9 ; 27 ; −81 ; 243

8. For A.P.:

   $T_1 = a = 1$

   $T_3 = a + 2d = 1 + 2d$

   $T_{13} = a + 12d = 1 + 12d$

   Thus the G.P. : 1 ; $1 + 2d$; $1 + 12d$

   now : by using the common ratio of the G.P. :

   $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

   $\therefore \frac{1+2d}{1} = \frac{1+12d}{1+2d}$

   $\therefore (1+2d)^2 = 1 + 12d$

   $\therefore 4d^2 + 4d + 1 = 1 + 12d$

   $\therefore 4d^2 - 8d = 0$

   $\therefore 4d(d-2) = 0$

   $\therefore d = 0$ *OR* $d = 2$

   n.a.

   the G.S. is: 1; 1 + 2(2); 1 + 12(2)

   1; 1 + 4; 1 + 24

   1; 5; 25

9. $T_n = ar^{n-1} \to 1 \cdot (2)^{n-1} > 1\,000$

   $\therefore \frac{2^n}{2} > 1\,000$

   $\therefore 2n > 2\,000$

   $\therefore n > \frac{\log 2\,000}{\log 2}$

   $\therefore n > 10{,}96...$

   $\therefore n = 11$

   So $T_{11} > 1000$

10. $T_5 = T_3 + 60$ *and* $T_2 + T_3 = 30.$

    $\therefore ar^4 - ar^2 = 60$ *and* $ar + ar^2 = 30.$

    So : $ar^2(r^2 - 1) = 60 \ldots (1)$ *and* $ar(1 + r) = 30 \ldots (2)$

    Now (1) ÷ (2): $\frac{ar^2(r^2-1)}{ar(1+r)} = \frac{60}{30}$

    $\therefore \frac{ar^2(r-1)(r+1)}{ar(r+1)} = 2$

    $\therefore \frac{r(r-1)}{1} = 2$

    $\therefore r^2 - r - 2 = 0$

    $\therefore (r-2)(r+1) = 0$

    $\therefore r = 2$ only since $r \neq -1$

    Now: $a(2) + a(4) = 30 \to 6a = 60 \to a = 10$

    The sequence : 10; 20; 40;.....

11. $\Rightarrow T_1 + T_2 = 12$ and $T_6 = 16T_{10}$

    $a + ar = 12$ and $ar^5 = 16\,ar^9$

    $\therefore \frac{ar^9}{ar^5} = \frac{1}{16}$

    $\therefore r^4 = \frac{1}{16}$

    $\therefore r = \pm\frac{1}{2}$

    now for $a(1 - r) = 12$:

    $a = \frac{12}{1+r}$

    $\therefore$ IF $r = -\frac{1}{2}$, THEN $a = \frac{12}{1-\frac{1}{2}} = \frac{12}{\frac{1}{2}} = 24$

    $\therefore$ IF $r = \frac{1}{2}$, THEN $a = \frac{12}{1+\frac{1}{2}} = \frac{12}{\frac{3}{2}} = \frac{24}{3} = 8$

    thus the sequences : a; ar; ar^2; …

if $r = \frac{1}{2}$	:	8 ; 4 ; 2 ; …………
if $r = -\frac{1}{2}$	:	24 ; −12 ; 6 ; ………..

Lesson 4

1. $\Rightarrow a + ar^2 = 280$ and $ar^4 + ar^5 = 4\,375$

$a(1 + r^2) = 280$ and $a(r^4 + r^5) = 4\,375$

$\underbrace{\therefore a} = \frac{280}{(1 + r^2)}$ and $a = \frac{4\,375}{r^4 + r^5}$ $r \neq 0; r \neq -1$

$\therefore \frac{280}{r(r + 1)} = \frac{4\,375}{r^4(1 + r)}$

$\therefore \frac{r^4(1 + r)}{r(1 + r)} = \frac{4\,375}{280}$

$\therefore r^3 = \frac{125}{8}$

$\therefore r = \frac{5}{2}$

then: $a = \frac{280}{\frac{5}{2} + \frac{25}{4}}$

$\therefore a = \frac{280}{\frac{10}{4} + \frac{25}{4}}$

$\therefore a = \frac{280}{\frac{35}{4}}$

$\therefore a = \frac{280 \cdot 4}{35}$

$\therefore a = \frac{1\,120}{35}$

$\therefore a = 32$

$\therefore T_4 = ar^3 = 32 \cdot \left(\frac{5}{2}\right)^3 = 32 \cdot 125$

2. $\Rightarrow S_n > 3\,000$

$a = 15$

$r = \frac{-30}{15} = -2$ ($\therefore r < 1$)

$n = ?$ $\therefore$ for $r < 1$: $Sn = \frac{a(1 - r^n)}{1 - r} > 3\,000$

$\therefore 3\,000 = \frac{15(1 - (-2)^n)}{1 + 2}$

$\therefore \frac{3\,000}{5} = (1 - (-2)^n)$

$\therefore 600 - 1 = -(-2)^n$

$\therefore 599 = -(-2)^n$

by inspection n must be odd.

if we solve $2^n = 599$

then: $n = \frac{\log 599}{\log 2}$

$n = 9{,}226412193$

thus $n = 10$ for $2^n = 599$

but n must be odd

thus $n = 11$.

3. $a = (a + 1)$ and $r = r^2$

a) $\therefore T_4 = ar^3 = (a + 1)(r^2)^3$

$\therefore T_4 = (a + 1)r^6$

b) $\therefore T_n = ar^{n-1}$

$\therefore T_n = (a + 1)(r^2)^{n-1}$

$\therefore T_n = (a + 1)r^{2n-2}$

c) $T_{2n} = ar^{2n-1}$

$\therefore T_{2n} = (a + 1)(r^2)^{2n-1}$

$\therefore T_{2n} = (a + 1)r^{4n-2}$

d) $S_n = \frac{a(r^n - 1)}{r - 1}$

$\therefore S_{n+1} = \frac{(a + 1)((r^2)^{n+1} - 1)}{r^2 - 1}$

$\therefore S_{n+1} = \frac{(a + 1)(r^{2n+2} - 1)}{r^2 - 1}$

4. $T_n = ar^{n-1}$

$\therefore \frac{1}{9} = 81 \cdot \left(\frac{1}{3}\right)^{n-1}$

$\therefore \left(\frac{1}{3}\right)^6 = \left(\frac{1}{3}\right)^{n-1}$

$\therefore 6 = n - 1$

$\therefore n = 7$

now:

$S_n = \frac{a(1 - r^n)}{1 - r}$

$\therefore S_7 = \frac{81\left(1 - \left(\frac{1}{3}\right)^7\right)}{1 - \frac{1}{3}}$

$\therefore S_7 = 121\frac{4}{9}$

5. a) $a = 42$ $T_5 = 1640{,}625$

$\therefore ar^4 = 1640{,}625$

$\therefore 42r^4 = 1640{,}625$

$\therefore r^4 = 39.0625$

$\therefore r = \pm 2{,}5$

$\therefore T_3 = ar^2 = 43(\pm 2{,}5)^2 = 262{,}5$

b) $S_{10} = ?$

$n = 10$

$r = \pm 2{,}5$

$a = 42$ for $r = 2{,}5$: $S_n = \frac{a(r^n - 1)}{r - 1}$; $r > 1$

$S_{10} = \frac{42((2{,}5)^{10} - 1)}{2{,}5 - 1}$

$S_{10} = 267\,000{,}8086$

for $r = -2{,}5$: $S_n = \frac{a(1 - r^n)}{1 - r}$ for $r < 1$

$S_{10} = \frac{42(1 - (-2{,}5)^{10})}{1 + 2{,}5}$

$S_{10} = -125156{,}629$

6. For 18 + 28+ 38+ ………… + 188:

This is clearly an A.S. with

$a = 14$

$d = 10$

$T_n = 188$

$Sn = ?$

$n = ?$ $Tn = a + (n - 1)d$

$188 = 18 + (n - 1)10$

$170 = 10(n - 1)$

$17 = n - 1$

$n = 18$

$S_n = \frac{n}{2}[a + T_n]$

now for S_{18}: $S_{18} = \frac{18}{2}[14 + 188]$

$S_{18} = 1\ 854$

for the sequence : 1 020 + 510 + 255 + ………. + 1,9921875

$a = 1\ 020$

$r = \frac{1}{2}$

$n = ?$

$T_n = 1,9921875$

$T_n = ar^{n-1}$

$\therefore 1,9921875 = (1\ 020)\left(\frac{1}{2}\right)^{n-1}$

$\therefore 0,001953125 = (0,5)^{n-1}$

$\therefore n - 1 = \frac{\log 0,001953125}{\log 0,5}$

$\therefore n - 1 = 9$

$\therefore n = 10$

now $S_n = \frac{a(1-r^n)}{1-r}; r < 1$

$\therefore S_{10} = \frac{1\ 020\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$

$= 2038,007813$

thus: $\frac{S_{18}}{S_{10}} = \frac{1\ 854}{2038,007813}$

$= 0,90971192$

7. The sequence: $\sum_{n=1}^{k} 3\cdot 2^{6-n} = 3.2^5 + 3.2^4 + 3.2^3 + \ldots\ldots\ldots\ldots$

$= 96 + 48 + 24 + \ldots\ldots\ldots\ldots$

Thus: $a = 96$; $r = \frac{1}{2}$; $n = k$; $S_n = \frac{765}{4}$

Since $|r| < 1$ $S_n = \frac{a(1-r^n)}{1-r}$

$\frac{765}{4} = \frac{96\left(1-\left(\frac{1}{2}\right)^k\right)}{\frac{1}{2}}\cdot\frac{2}{2}$

$\therefore 765 = 96\left(1-\left(\frac{1}{2}\right)^k\right)\cdot 4$

$\therefore 765 = 768 - 768\left(\frac{1}{2}\right)^k$

$\therefore -3 = -768\left(\frac{1}{2}\right)^k$

$\therefore 0,0039062 = (0,5)^k$

$\therefore k = \frac{\log 0,0039062}{\log 0,5} = 7,999999999 = 8$

$\therefore k = 8.$

8. If we look carefully, we will see that all the first numbers in the products are odd natural numbers, and all the second factors are even .

The general pattern is therefore: $\frac{1}{(2r-1)(2r)}$

To find n in $\sum_{r=1}^{n}\ldots$, we say $2r - 1 = 99$ or $2r = 100$

$\therefore 2r = 100$ or $r = 50$

$* \ r = 50$

$\therefore \sum_{r=1}^{50}\frac{1}{(2r-1)(2r)}$

9. The general pattern is that the first factor in each term is one less than the second:

Thus the general term is: $\frac{1}{n(n+1)}$

To find the number of terms: $n = 99$ or $n + 1 = 100 \rightarrow n = 99$

$\therefore \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots + \frac{1}{99.100} = \sum_{r=1}^{99}\frac{1}{r(r+1)}$

10. There are two factors in each term . If we only consider the first factor of every term:

1 + 4 + 7 + ………. + 100 , This is an AP with:

$a = 1$

$d = 3$

$T_n = 100$

Thus: $T_n = a + (n-1)d$

$\therefore 100 = 1 + (n-1)3$

$\therefore 33 = n - 1$

$\therefore n = 34$

Now with $T_n = 1 + 3(n-1) = 3n - 2$ and $n = 34$:

$1 + 4 + 7 + \ldots + 100 = \sum_{n=1}^{34}(3n-2)$

For each second factor in each term : 2 + 5 + 8 + ………. + 101

This is also an AP with: $a = 2$, $d = 3$ and $T_n = 101$

Thus: $T_n = a + (n-1)d$

$\therefore 101 = 2 + (n-1)3$

$\therefore 33 = n - 1$

$\therefore n = 34$

Now with $n = 34$ and $T_n = 2 + 3n - 3 = 3n - 1$:

$2 + 5 + 8 + \ldots + 101 = \sum_{n=1}^{34}(3n-1)$

Finally: $1.2 + 4.5 + 7.8 + \ldots + 100.101 = \sum_{n=1}^{34}(3n-2)(3n-1)$

11. $\sum_{k=1}^{n}\frac{4}{3}\left(\frac{3}{2}\right)^{k-1} = \frac{665}{24}$

$a = \frac{4}{3}$ and $r = \frac{3}{2}$

So: $S_n = \frac{a(r^n - 1)}{r-1}$

$\therefore \frac{665}{24} = \frac{\frac{4}{3}\left(\left(\frac{3}{2}\right)^n - 1\right)}{\frac{3}{2}-1}\cdot\frac{6}{6} = \frac{8\left(\left(\frac{3}{2}\right)^n - 1\right)}{3}$

$\therefore 3 \times 665 = 24 \times 8\left(\left(\frac{3}{2}\right)^n - 1\right)$

$\therefore \frac{665}{64} = \left(\frac{3}{2}\right)^n - 1$

$\therefore \frac{729}{64} = \left(\frac{3}{2}\right)^n$

$\therefore \left(\frac{3}{2}\right)^6 = \left(\frac{3}{2}\right)^n$

$\therefore n = 6$

Lesson 5

1. $118 + 12 + \left(\frac{5}{8}\right)(12) + \left(\frac{5}{8}\right)^2(12) \ldots 130 + S_\infty$

$a = \frac{15}{2} \quad r = \frac{5}{8}$

$= 130 + \frac{7,5}{1-\frac{5}{8}}$

$= 130 + \frac{15}{2} \times \frac{8}{3}$

$= 130 + 5\cdot 4 = 150$

2. a) $a + ar + ar^2 + \ldots = 3$

$\frac{a}{1} - r = 3$

$\therefore a = 3(1 - r) \ldots (1)$

b) $a^2 + a^2r^2 + a^2r^4 + \ldots = 3$

$\frac{a^2}{1} - r^2 = 3$

$\therefore a^2 = 3\,(1 - r^2) \ldots (2)$

$(1) \to (2)\ [3(1 - r)]^2 = 3\,(1 - r^2)$

$\therefore 9(1 - r)^2 = 3(1 - r)(1 + r)$

$\therefore 3(1 - r)[3(1 - r) - (1 + r)] = 0$

$\therefore 1 - r = 0$ or $3 - 3r - 1 - r = 0$

$\therefore r = 1 \qquad \therefore 2 - 4r = 0$

$r = \frac{1}{2}$

But $-1 < r < 1$ to converge

So $r = \frac{1}{2}$ only

and $a = 3\left(1 - \frac{1}{2}\right) = \frac{3}{2}$

3. $A_1 + A_2 + A_3 + A_4$ . . . . where A is area.

$A_1 = 4^2 = 16$

$A_2 = EF^2$ and $EF = \sqrt{8}$

$\therefore A_2 = \sqrt{8}^2 = 8$

$A_3 = PQ^2 \Rightarrow PQ = \sqrt{\frac{8}{4} + \frac{8}{4}}$

$\therefore A_3 = 4$

$\therefore PQ^2 = 4$

Sequence: $16 + 8 + 4 + \ldots$

$S_\infty = \frac{a}{1 - r} \quad a = 16 \qquad r = \frac{1}{2}$

$= \frac{16}{\frac{1}{2}} = 32$

4. To converge : $|r| < 1$

$-1 < \frac{6x}{3x^2} < 1$

$\therefore -1 < \frac{2}{x} < 1$

$\therefore -1 < \frac{1}{x} < \frac{1}{2}$

$\therefore x > 2$ or $x < -2$

now: $a = 3x^2$ and $r = \frac{2}{x}$

$\therefore S_\infty = \frac{a}{1 - r}$

$\therefore S_\infty = \frac{3x^2}{1 - \frac{2}{x}}$

$\therefore S_\infty = \left(\frac{3x^2}{1 - \frac{2}{x}}\right)\cdot\left(\frac{x}{x}\right)$

$\therefore S_\infty = \frac{3x^3}{x - 2}$

5. $a = 2 \qquad S_\infty = 12$

$S_\infty = \frac{a}{1 - r} \Rightarrow \frac{2}{1 - r} = 12$

$\Rightarrow 2 = 12 - 12r$

$12r = 10$

$r = \frac{5}{6}$

6. $a = 4 \quad r = -\frac{3}{4}$

$S_\infty = \frac{a}{1 - r}$

$= \frac{4}{1\frac{3}{4}}$

$= \frac{4}{1} \times \frac{4}{7}$

$= \frac{16}{7} = 2\,\frac{2}{7}$

7. $S_\infty = \frac{3}{2}$ and $S_3 = \frac{14}{9}$

$\frac{a}{1 - r} = \frac{3}{2} \ldots (1)$ and $\frac{a\,(1 - r^3)}{1 - r} = \frac{14}{9} \ldots (2)$

(since $r < 1$)

$(1) \to (2)$: $\frac{3}{2}\,(1 - r^3) = \frac{14}{9}$

$\therefore 1 - r^3 = \frac{28}{27}$

$\therefore r^3 = 1 - \frac{28}{27}$

$= -\frac{1}{27}$

$\therefore r = \frac{-1}{3}$

Now $a = \frac{3}{2}\left(1 - \left(-\frac{1}{3}\right)\right)$

$= \frac{3}{2}\left(\frac{4}{3}\right)$

$a = 2$

$\therefore 2; -\frac{2}{3}; \frac{2}{9}; \ldots$

8. $\sum_{n=1}^{\infty} 3\cdot 2^{6-n}$

$T_1 = 3\cdot 2^5 = 96$

$T_2 = 3\cdot 2^4 = 48$

$r = \frac{T_2}{T_1} = \frac{48}{96} = \frac{1}{2}$

$S_\infty = \frac{a}{1 - r}$

$= \frac{96}{1 - \frac{1}{2}} \times \frac{2}{2}$

$S_\infty = 192$

9. $0{,}27272727\ldots = 0{,}27 + 0{,}0027 + 0{,}000027 + \ldots$

$= \frac{27}{100} + \frac{27}{10\,000} + \frac{27}{1\,000\,000} + \ldots$

$a = \frac{27}{100}$ and $r = \frac{1}{100}$ such that $|r| < 1$

$S_\infty = \frac{a}{1 - r}$

$\therefore S_\infty = \frac{\frac{27}{100}}{1 - \frac{1}{100}}$ thus $S_\infty = \frac{3}{11} = 0.27272727...$

$\therefore S_\infty = \frac{27}{99}$

10. $\frac{a}{1 - r} = 9$ and $ar = 2$.

Now: $a = \frac{2}{r} \to \frac{\frac{2}{r}}{1 - r} = 9.$

$\therefore \frac{2}{r - r^2} = 9$

$\therefore 9r^2 - 9r + 2 = 0$

$\therefore (3r - 2)(3r - 1) = 0$

$\therefore r = \frac{1}{3}$ or $r = \frac{2}{3}$

So from $a = \frac{2}{r}$: $a = \frac{2}{\frac{1}{3}} = 6$ and $a = \frac{2}{\frac{2}{3}} = 311.$ a) $0{,}3 = 0{,}33333\ldots$ to infinity

$= 0{,}3 + 0{,}03 + 0{,}003 + \ldots$ to infinity

$a = 0{,}3$

$r = \frac{0{,}03}{0{,}3} = \frac{3}{30} = \frac{1}{10} = 0{,}1$

$S_\infty = \frac{a}{1 - r} = \frac{0{,}3}{1 - 0{,}1} = \frac{3}{9} = \frac{1}{3}$

$\therefore 0{,}\dot{3} = \frac{1}{3}$

b) 0,657 = 0,657575757 … to infinity

$= 0,6 + 0,057 + 0,00057$ … to infinity

$= 0,6 + S_\infty$ of a convergent series with $a = 0,057 \quad r = 0,01$

$= 0,6 + \frac{a}{1-r}$

$= 0,6 + \frac{0,057}{0,99}$

$= \frac{6}{10} + \frac{57}{990}$

$= \frac{594 + 57}{990}$

$= \frac{651}{990}$

$= \frac{217}{330}$

$\therefore 0,6\dot{5}\dot{7} = \frac{217}{330}$

Lesson 6

1. 5 8 13 20 29 ....

+3 +5 +7 +9 → First order difference is not constant

2 2 2 → Second order difference is constant

So for the general rule:

	T_1	T_2	T_3	T_4	T_5
n^2	1	4	9	16	25
Want	5	8	13	20	29
Short	4	4	4	4	4

$T_n = n^2 + 4; n \in \mathbb{N}$

The recursive relationship:

5

+3 +5 +7 +9

$T_1 = 5$

$T_2 = T_1 + 3$

$T_3 = T_2 + 5$

$T_4 = T_3 + 7$

$T_5 = T_4 + 9$

Focusing on $3 + 5 + 7 + 9 + \ldots = 3 + (n-1)(2) = 2n + 1$

So: $\begin{cases} T_{n+1} = T_n + 2n + 1 \\ T_1 = 5; n \in \mathbb{N} \end{cases}$ or $\begin{cases} T_n = T_{n-1} + 2n - 1 \\ T_1 = 5; n \in \mathbb{N} \end{cases}$

2. 1 2 9 22 41 ....

+1 +7 +13 +19 → First order difference is not constant

6 6 6 → Second order difference is constant

So for the general rule:

	T_1	T_2	T_3	T_4	T_5
$3n^2$	3	12	27	48	75
Want	1	2	9	22	41
Short	–2	–10	–18	–26	–34

Focus on –2 ; –10 ; –18 ; –26 ; …= $-2 + (n-1)(-8) = 6 - 8n$

$T_n = 3n^2 + 6 - 8n; n \in \mathbb{N}$

The recursive relationship:

1

+1 +7 +13 +19

$T_1 = 1$

$T_2 = T_1 + 1$

$T_3 = T_2 + 7$

$T_4 = T_3 + 13$

$T_5 = T_4 + 19$

Focusing on $1 + 7 + 13 + 19 + \ldots = 1 + (n - 1)(6) = 6n - 5$

So: $\begin{cases} T_{n+1} = T_n + 6n - 5 \\ T_1 = 1; n \in \mathbb{N} \end{cases}$ or $\begin{cases} T_n = T_{n-1} + 6n - 11 \\ T_1 = 1; n \in \mathbb{N} \end{cases}$

3. –2 2 9 19 32 ....

+4 +7 +10 +13 → First order difference is not constant

3 3 3 → Second order difference is constant

So for the general rule:

	T_1	T_2	T_3	T_4	T_5
$\frac{3}{2}n^2$	$\frac{3}{2}$	$\frac{12}{2}$	$\frac{27}{2}$	$\frac{48}{2}$	$\frac{75}{2}$
Want	$\frac{-4}{2}$	$\frac{4}{2}$	$\frac{18}{2}$	$\frac{38}{2}$	$\frac{64}{2}$
Short	$\frac{-7}{2}$	$\frac{-8}{2}$	$\frac{-9}{2}$	$\frac{-10}{2}$	$\frac{-11}{2}$

$T_n = \frac{3}{2}n^2 - \frac{n+6}{2}; n \in \mathbb{N}$

The recursive relationship:

$T_1 = -2$

$T_2 = T_1 + 4$

$T_3 = T_2 + 7$

$T_4 = T_3 + 10$

$T_5 = T_4 + 13$

–2

+4 +7 +10 +13

Focusing on $4 + 7 + 10 + 13 + \ldots$

$= 4 + (n - 1)(3) = n + 1$

So: $\begin{cases} T_{n+1} = T_n + 3n + 1 \\ T_1 = -2; n \in \mathbb{N} \end{cases}$ or $\begin{cases} T_n = T_{n-1} + 3n - 2 \\ T_1 = -2; n \in \mathbb{N} \end{cases}$

TIPS FOR THE TEACHER

Lesson 1

- It is important that the learners are able to identify sequences and make formulas.
- Try and use as many real-life examples as possible.
- Include the quadratic equations covered in Grade 11 for testing this section.

Lesson 2

- It is important that the learners are able to identify sequences and make formulas.
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Lesson 4

- It is important that the learners are able to identify sequences and make formulas.
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Lesson 5

- It is important that the learners are able to identify sequences and make formulas.
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- Include the quadratic equations covered in Grade 11 for testing this section.

Lesson 6

- It is important that the learners are able to identify sequences and make formulas.
- Try and use as many real-life examples as possible.
- Include the quadratic equations covered in Grade 11 for testing this section.