SEQUENCES AND SERIES

Sigma notation and the geometric series

Lesson 4

Learning Outcomes and Assessment Standards

Learning outcome 1: Number and number relationships
Assessment standard 12.1.3

- Identify and solve problems involving number patterns, including but not limited to arithmetic and geometric sequences and series.
- Correctly interpret sigma notation.
- Prove and correctly select the formula for and calculate the sum of series, including:

$$\sum_{i=1}^{n} 1 = n; \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}; \quad \sum_{i=1}^{n} a + (i-1)d = \frac{n}{2}[2a + (n-1)d]$$

$$\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n - 1)}{r - 1};\ r \neq 1; \quad \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r};\ -1 < r < 1$$

- Correctly interpret recursive formulae: (e.g. $T_{n+1} = T_n + T_{n-1}$)

Overview

In this lesson you will:

- put geometric series into sigma notation
- derive a formula for the sum of a geometric series
- apply this formula to mathematical problems
- apply this formula to real-life examples.

Lesson

DVD

What does the following mean? $\sum_{k=1}^{5} \frac{1}{2}(2)^{k-1}$

The sum of the first five terms in the sequence with rule $T_k = \frac{1}{2}(2)^{k-1}$

So: $\frac{1}{2}(2)^0 + \frac{1}{2}(2) + \frac{1}{2}(2)^2 + \frac{1}{2}(2)^3 + \frac{1}{2}(2)^4$

$= \frac{1}{2} + 1 + 2 + 4 + 8 = 15\frac{1}{2}$

You may need to put sequences into sigma notation.

Example 1

Try $12 + 6 + 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8}$

Solution

$a = 12;\ r = \frac{1}{2};\ n = 6$ terms

$T_k = ar^{k-1} = 12\left(\frac{1}{2}\right)^{k-1}$

$= 12 \cdot \left(\frac{1}{2}\right)^{k} \cdot \left(\frac{1}{2}\right)^{-1}$

$= 12 \cdot 2 \cdot \left(\frac{1}{2}\right)^{k}$

$= 24 \cdot 2^{-k}$

$\therefore \sum_{k=1}^{6} 24 \cdot 2^{-k}$ or $\sum_{k=1}^{6} \frac{(-3)^k}{2^k}$

Example 2

$-1 + 3 - 9 + 27 - 81 + \ldots$ to n terms

Solution

The general rule: a = –1; r = –3

$\therefore T_k = ar^{k-1} = (-1)(-3)^{k-1}$

$= (-1)(-3)^k\left(-\frac{1}{3}\right)$

$= \frac{1}{3}(-3)^k$

Example 3

1(2) + (2)(5) + 4(8) + 8(11) … to n terms

Solution

The sum of a geometric series multiplied by an arithmetic series

1(2) + 2(5) + 4(8) + 8(11) + … n terms

We have a product of two sequences:

Sequence 1: 1; 2; 4; 8; … which is a geometric sequence with $a = 1$ and $r = 2$.

So $T_n = ar^{n-1} = (1)(2)^{n-1} = 2^{n-1} = \frac{2^n}{2}$

Sequence 2: 2; 5; 8; 11; … which is an arithmetic sequence with $a = 2$ and $d = 3$.

So $T_n = a + (n-1)d = 2 + (n-1)(3) = 3n - 1$

So the sum to n terms: $\sum_{k=1}^{n}(2)^{k-1}(3k-1)$

Deriving a formula for the sum of a geometric series

(S_n): $S_n = a + ar + ar^2 + \ldots ar^{n-3} + ar^{n-2} + ar^{n-1}$

$(S_n \times r)$: $rS_n = ar + ar^2 + \ldots\ldots ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n$

$S_n - rS_n = a \qquad - ar^n$

$S_n(1 - r) = a - ar^n$

$S_n = \frac{a(1-r^n)}{1-r}$ or $S_n = \frac{a(r^n-1)}{r-1}$; $r \neq 1$

We use $S_n = \frac{a(1-r^n)}{1-r}$ if $r < 1$ and $S_n = \frac{a(r^n-1)}{r-1}$ if $r > 1$.

Example 1

Determine $\sum_{r=1}^{13} 6\left(-\frac{1}{3}\right)^{r-1}$

Solution

$a = 6$; $r = -\frac{1}{3}$ use $Sn = \frac{a(1-r^n)}{1-r}$ since $r < 1$

$$Sn = \frac{6\left[1-\left(-\frac{1}{3}\right)^{13}\right]}{1-\left(-\frac{1}{3}\right)}$$

$$Sn = \frac{6\left[1-\left(-\frac{1}{3}\right)^{-13}\right]}{1+\frac{1}{3}}$$

$$= \frac{6(1+3^{13})}{\frac{4}{3}}$$

= 7174458 (Put this into your calculator)

Example 2

Determine n if $\sum_{k=1}^{n}(-8)\left(\frac{1}{2}\right)^{k-1} = -15\frac{3}{4}$

Solution

$a = -8$; $r = \frac{1}{2}$ use $Sn = \frac{a(1 - r^n)}{1 - r}$

$$-\frac{63}{4} = -8\left[\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\right]$$

$$\frac{63}{32} = \frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}}$$

$$\frac{63}{64} = 1 - \left(\frac{1}{2}\right)^n$$

$$\left(\frac{1}{2}\right)^n = \frac{1}{64}$$

$$\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^6$$

$$n = 6$$

Example 3

A drill reaches a depth of 120 m on the first day. Thereafter it drills $\frac{2}{3}$ of the depth of the previous day on each successive day. If the driller is paid R40 per metre, how much will he have earned by the end of the 5th day?

Solution

$a = 120$ $r = \frac{2}{3}$ $n = 5$ use $Sn = \frac{a(1 - r^n)}{1 - r}$ since $r < 1$

$S_5 = \frac{120\left[1 - \left(\frac{2}{3}\right)^5\right]}{\frac{1}{3}}$ Use your calculator

$S_5 = 312{,}59259 \ldots \left(\text{or } \frac{8\,440}{27}\right)$

So at R40 per metre he will have earned $\left(\frac{8\,440}{27}\right) \times 40 = \text{R}12\ 503{,}70$

Activity

1. The sum of the second and the third terms of a G.P. is 280, and the sum of the 5^{th} and the 6^{th} terms is 4 375. Find the 4^{th} term.

2. How many terms of the geometric sequence $15 - 30 + 60 - 120 + \ldots$ must be taken for the sum to exceed 3 000?

3. The first term of a G.P. is $(a + 1)$, and the common ratio is r^2 . Write down an expression for

   a) T_4

   b) T_n

   c) T_{2n}

   d) S_{n+1}

4. Determine the sum of the G.S. $81 + 27 + 9 + \ldots + \frac{1}{9}$

5. Given the geometric sequence 42; … ; …; …; 1 640,625

   By using appropriate formulae, calculate :

   a) the third term

   b) the sum of the first ten terms of the sequence

6. Evaluate: $\dfrac{18 + 28 + 38 + \ldots + 188}{1\,020 + 510 + 255 + \ldots + 1.9921875}$

7. Determine k if $\sum_{n=1}^{k} 3 \cdot 2^{6-n} = \frac{765}{4}$

8. Express $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots + \frac{1}{99.100}$ using sigma notation

9. Express $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots + \frac{1}{99.100}$ in sigma notation.

10. Write the following in sigma notation: 1.2 + 4.5 + 7.8 + ………. + 100.101

11. Determine n if: $\sum_{k=1}^{n} \frac{4}{3}\left(\frac{3}{2}\right)^{k-1} = \frac{665}{24}$