Lesson 5

SEQUENCES AND SERIES

Sum to infinity of a converging geometric series

Learning Outcomes and Assessment Standards

Learning outcome 1: Number and number relationships
Assessment standard 12.1.3

- Identify and solve problems involving number patterns, including but not limited to arithmetic and geometric sequences and series.
- Correctly interpret sigma notation.
- Prove and correctly select the formula for and calculate the sum of series, including:

$$\sum_{i=1}^{n} 1 = n; \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}; \quad \sum_{i=1}^{n} a + (i-1)d = \frac{n}{2}[2a + (n-1)d]$$

$$\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n - 1)}{r - 1}\ r \neq 1; \quad \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r};\ -1 < r < 1$$

- Correctly interpret recursive formulae: (e.g. $T_{n+1} = T_n + T_{n-1}$)

Overview

In this lesson you will:

- investigate what happens to different series when we add them indefinitely
- understand what is meant by diverge and converge
- understand what happens when a series converges
- develop a formula for S_∞
- solve problems involving sums to infinity.

Lesson

DVD

Let's investigate an arithmetic series with a positive constant difference.

$2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 \ldots$ so $S_\infty = ?$ (very big)

The series diverges to a very large number. So $S_\infty = \infty$

Let's investigate an arithmetic series with a negative constant difference.

$3 - 1 - 5 - 9 - 13 - 17 - 21 - 25 - 29 - 33 - \ldots$ so $S_\infty = ?$ (very small)

The series diverges to a very small negative number. So $S_\infty = -\infty$.

Let's investigate a geometric series with $r > 1$.

$2 + 4 + 8 + 16 + 32 + 64 + 128 + ...$ so $S_\infty = ?$ (very big, very quick)

The series diverges.

Let's investigate a geometric series with $-1 < r < 1$

$16 + 8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots$

$S_\infty = 32$

We have an answer. We say the series CONVERGES.

A geometric series converges if $-1 < r < 1$

Example

$(2x + 4) + (2x + 4)^2 + (2x + 4)^3 + \dots$ [For $-1 < r < 11$]

For which values of x will the series converge?

Solution

$-1 < \frac{T_2}{T_1} < 1$ (to converge)

$-1 < 2x + 4 < 1$

$-5 < 2x < -3$

$\frac{-5}{2} < x < \frac{-3}{2}$

Deriving a formula for the sum to infinity:

If $-1 < r < 1$

$S_n = \frac{a(1 - r^n)}{1 - r}$

Now since $-1 < r < 1$, $r^n \rightarrow 0$ as $n \rightarrow \infty$

So $S_\infty = \frac{a(1 - r^n)}{1 - r}$

$= \frac{a(1 - 0)}{1 - r}$

$= \frac{a}{1 - r}$

Example $16 + 8 + 4 + \dots$

$a = 16 \quad r = \frac{1}{2}$

So $S_\infty = \frac{16}{\frac{1}{2}} = 32$

Real life examples

1. Under ideal conditions a tree in the nursery grows 50 cm in the first year and then a further 100 cm the next year. It is then replanted and each year it grows by $\frac{2}{3}$ of the previous year's growth. What is the highest that the tree will grow to?

Solution

$50 + 100 + 100\left(\frac{2}{3}\right) + 100\left(\frac{2}{3}\right)^2 + \dots$

$= 50 + 100\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots\right]$

$= 50 + 100 \times S_\infty$

So for S_∞: $a = 1 \quad r = \frac{2}{3}$

$\therefore S_\infty = \frac{a}{1 - r} = \frac{1}{1 - \frac{2}{3}} \times \frac{3}{3}$

$= \frac{3}{3 - 2}$

$= 3$

$\therefore$ Total length of the tree is $50 + 100 \times 3 = 350$ cm.

2. A man stands on a wall 6 m high and drops a bouncing ball. Each bounce is nine tenths as high as the previous bounce. What is the distance the ball bounces before coming to rest?

Solution

$6 + 6\left(\frac{9}{10}\right)(2) + 6\left(\frac{9}{10}\right)^2(2) + 6\left(\frac{9}{10}\right)^3(2) + \dots$

$= 6 + 6\left(\frac{9}{10}\right)(2)\left[1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^3 + \dots\right]$

$6 + 6\left(\frac{9}{10}\right)(2) \times S_\infty$

For S_∞: $a = 1$ $r = \frac{9}{10}$

$\therefore S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{9}{10}} \times \frac{10}{10}$

$= \frac{10}{10-9}$

$\therefore S_\infty = 10$

$\therefore$ Distance $= 6 + 6\left(\frac{9}{10}\right)(2)(10)$

$= 6 + 108$

$= 114$ m

3. A little frog is 5 metres away from the pond. He can jump 1,2 m and each jump from then on is three quarters of the distance of the previous jump. Will he reach the pond before he is too exhausted and can no longer jump?

Solution

$1{,}2 + 1{,}2\left(\frac{3}{4}\right) + 1{,}2\left(\frac{3}{4}\right)^2 + 1{,}2\left(\frac{3}{4}\right)^3 + \ldots$

$= 1{,}2 + 1{,}2\left(\frac{3}{4}\right)(2)\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \ldots\right]$

$= 1{,}2 + 1{,}2\left(\frac{3}{4}\right) \times S_\infty$

For S_∞: $a = 1$ $r = \frac{3}{4}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{3}{4}} \times \frac{4}{4} = 4$

$\therefore$ Distance jumped: $= 1{,}2 + 1{,}2\left(\frac{3}{4}\right) \times 4$

$= 1{,}2 + 3{,}6$

$= 4{,}8$ m

So he is still 0,2 m away from the pond.

4. In a recycling bin, the quantity of tin recovered during the first cycle is 300 kg, during the 2nd cycle 240 kg, during the 3rd cycle 129 kg, and so on.

   a) Show that the recycling process represents a geometric sequence.

   b) Calculate the total quantity recovered during the first 8 cycles.

   c) If the process is continually repeated, calculate the total mass that would be recovered.

   d) In order for the process to be economical at least 10 kg must be recovered in a cycle. What is the greatest number of cycles for this process to be economical?

Solutions

a) 300; 240; 192; …

$r = \frac{240}{300} = \frac{4}{5}$ $\quad r = \frac{192}{240} = \frac{4}{5}$

b) $a = 300$ $\quad r = \frac{4}{5}$

$S_n = \frac{a(1-r^n)}{1-r}$ $\quad S_n = \frac{300\left(1-\left(\frac{4}{5}\right)^8\right)}{\frac{1}{5}} = 1\,248{,}3$ kg

c) $S_\infty = \frac{a}{1-r} = \frac{300}{\frac{1}{5}} = 1\,500$ kg

d) Find n if $T_n > 10$

$ar^{n-1} > 10$

$300\left(\frac{4}{5}\right)^{n-1} > 10$

$\left(\frac{4}{5}\right)^{n-1} > \frac{1}{30}$

So $n - 1 < \frac{\log\left(\frac{1}{30}\right)}{\log\left(\frac{4}{5}\right)}$

$n - 1 < 15,242 \ldots$

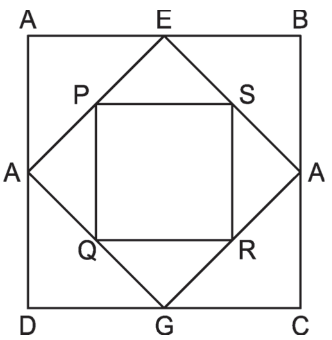
$n < 16, 242 \ldots$

$\therefore\ n = 16$

Activity

1. A certain plant reaches a height of 118 mm after one year under ideal conditions in a hothouse. During the next year, the height increases by 12 mm and in each successive year by five eighths of the previous year's growth. Show that the plant will never reach a height of more than 150 mm.
2. a) The sum to infinity of the series $a + ar + ar^2 + \ldots$, $|r| < 1$ is 3. Write down an equation containing a, r and 3.

   b) A second series is formed by squaring the terms of the series in (a). If the sum to infinity of this series is also 3, find the values of a and r.
3. ABCD is a square with sides 4 cm. E, F, G, H are mid-points of AD, AB, BC and CD respectively. A new square EFGH is so formed. Similarly a new square PQRS is formed, and so on. Find the sum of the areas of all the squares so formed.



4. Find the sum of the infinite G.P. $3x^2 + 6x + 12 + \frac{24}{x} + \ldots$ Also state the values of x for which the infinite sum exists.
5. Find r if $\sum_{n=1}^{\infty} 2r^{n-1} = 12$.
6. Find the sum to infinity of $4 - 3 + 2\frac{1}{4} \ldots$
7. In a geometric sequence, the sum to infinity is $1\frac{1}{2}$ and the sum of the first three terms is $\frac{14}{9}$. Determine the sequence.
8. Evaluate $\sum_{n=1}^{\infty} 3.2^{6-n}$
9. By first writing the recurring decimal 0,272727…… in the form $\frac{27}{100} + \frac{27}{10\,000} + \frac{27}{1\,000\,000} + \ldots$ write it as a vulgar fraction .
10. a) The sum to infinity of a G.P. is 9, and the second term is 2. Show that there are two such geometric progressions, and find the first term of each.

    b) Show that no G.P. has a sum to infinity of 1 and a second term of −2.
11. Write the following recurring decimals as proper fractions:

    a) $0,\dot{3}$

    b) $0,6\dot{5}\dot{7}$