CALCULUS (1)

Lesson 18

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and Algebra
Assessment standard 12.2.7(a)
Investigate and use instantaneous rate of change of a variable when interpreting models of situations:

- demonstrating an intuitive understanding of the limit concept in the context of approximating the rate of change or gradient of a function at a point
- establishing the derivative of the following functions from first principles
  $f(x) = b$; $f(x) = x^2$; $f(x) = \frac{1}{x}$; $f(x) = x$; $f(x) = x^3$
  and then generalise to the derivative of $f(x) = x^n$.

Overview

In this lesson you will:

- revise the average gradient studied in grade 11
- establish the gradient at a point by first principles
- understand the noun "the derivative" and the verb "to differentiate".

Lesson

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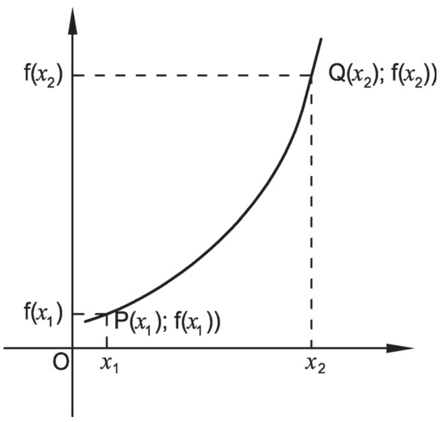
Definition of the average gradient (the average rate of change)

The average gradient between any two points on a curve is the gradient of the line which passes through these two points.

The average gradient between P and Q is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Example 1

If $f(x) = 2x^2 - 3$, find the average gradient between $x = -1$ and $x = -2$

Solution

Average gradient $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$f(x_2) = f(-2)$ $\quad$ $f(x_1) = f(-1)$

$= 2(4) - 3$ $\quad$ $= 2(1) - 3$

$= 5$ $\quad$ $= -1$

Average gradient $= \frac{5 - (-1)}{-2 - (-1)}$

$= \frac{6}{-1}$

$= -6$

Remember: The average gradient is the gradient between any two points on a curve.

The average gradient is also the average rate of change.

Example 2

The height of a stone, when thrown vertically into the air, is given by the equation $s(t) = 50t - 5t^2$ where $s(t)$ is distance in metres and t is time in seconds.

Find the average speed of the stone between 1 and 2 seconds.

Solution

Let $t_1 = 1$ and $t_2 = 2$

Average speed is $\frac{s(2) - s(1)}{2 - 1}$

$s(2) = 100 - 20$ $\qquad$ $s(1) = 50 - 5$

$= 80$ $\qquad$ $= 45$

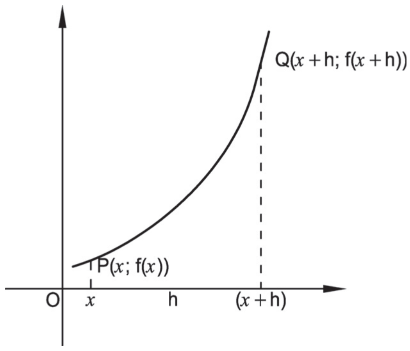
Average speed is $\frac{80 - 45}{1} = 35 \text{ m·s}^{-1}$

Remember:
Average speed is the total distance over the total time

The gradient at a point (the instantaneous gradient)

The average gradient between P and Q is

$$\frac{f(x + h) - f(x)}{x + h - x}$$
$$= \frac{f(x + h) - f(x)}{\text{h}}$$



What if we want the gradient at P?

Gradient is $\frac{0}{0}$. What's that?

We need to make a plan?

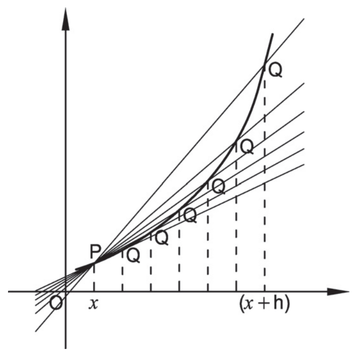
As the two points move closer and closer to each other, the value of h gets closer and closer to zero

$$h \to 0$$

The closer h gets to zero, the closer we are to the gradient of a curve at a point or the gradient of a tangent.

This is known as the gradient function. It is:

- The gradient of a curve at a point
- The gradient of a tangent
- The instantaneous rate of change
- The derivative.



We need new notation.

We use $\lim\limits_{h \to 0}$

This means the distance between the two points is very, very, very small (very close to 0).

Average gradient $= \frac{f(x + h) - f(x)}{h}$

Gradient of the tangent $= \lim\limits_{h \to 0} \frac{f(x + h) - f(x)}{h}$ (Instantaneous gradient or the derivative)

Notation for the derivative	
Function	**Derivative**
$f(x) = \ldots$ $y = ax^2 \ldots$ $s = 5t^2$ $x^2 - x - 1$	$f'(x) = \ldots$ $\frac{d}{dx}(x^2 - x - 1) = \ldots$ $\frac{ds}{dt} = \ldots$ $\frac{dy}{dx} = \ldots$

When we use $\lim\limits_{h \to 0} \frac{f(x + h) - f(x)}{h}$ to differentiate we say we are differentiating by <u>first principles</u>.

Example 1

If $f(x) = x^2$, find $f'(x)$ by first principles.

Solution

Firstly find the average gradient: $f(x) = x^2$

So $\quad f(x + h) = (x + h)^2 = x^2 + 2hx + h^2$

Thus: $$\frac{f(x + h) - f(x)}{h} = \frac{x^2 + 2hx + h^2 - x^2}{h}$$
$$= \frac{2hx + h^2}{h}$$
$$= \frac{(2x + h)\cancel{h}}{\cancel{h}}$$
$$= 2x + h$$

Now find $f'(x)$ by taking the limit:

$$\therefore\ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} 2x + h$$
$$= 2x$$

Example 2

Find the gradient of the tangent to $f(x) = -2x^2 + 1$ at $x = -1$ by first principles.

Solution

$$f(x) = -2x^2 + 1$$
$$f(x + h) = -2(x + h)^2 + 1$$
$$= -2(x^2 + 2hx + h^2) + 1$$
$$= -2x^2 - 4hx - 2h^2 + 1$$
$$\therefore\ \frac{f(x + h) - f(x)}{h} = \frac{-2x^2 - 4hx - 2h^2 + 1 + 2x^2 - 1}{h}$$
$$= \frac{-4hx - 2h^2}{h}$$
$$= -4x - 2h$$
$$\therefore\ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} -4x - 2h$$
$$= -4x$$
$$f'(-1) = -4(-1) = 4$$

Example 3

If $y = \frac{2}{x}$, find the gradient of the point $x = 4$ by first principles.

Solution

$$f(x) = \frac{2}{x}$$
$$f(x + h) = \frac{2}{x + h}$$
$$\therefore\ \frac{f(x + h) - f(x)}{h} = \frac{\frac{2}{x + h} - \frac{2}{x}}{h} \times \frac{x(x + h)}{x(x + h)}$$
$$= \frac{2x - 2(x + h)}{hx(x + h)}$$
$$= \frac{-2h}{hx(x + h)}$$
$$= -\frac{2}{x(x + h)}$$
$$\therefore\ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{-2}{x(x + h)}$$
$$= -\frac{2}{x^2}$$
$$f'(4) = -\frac{2}{16} = -\frac{1}{8}$$

Example 4

If $f(x) = 2$, find $f'(x)$ by first principles.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{2 - 2}{h}$$
$$= \lim_{h \to 0} \frac{0}{h}$$
$$= 0$$

Activity 1

1. If $f(x) = 1 - x - x^2$, find the average gradient between the points
   a) $x = 1$ and $x = -1$
   b) $x = -2$ and $x = -3$
2. If $f(x) = 1 - 2x^2$, find the average gradient between the points
   a) $x = 1$ and $x = 2$
   b) $x = -1$ and $x = -2$
3. If $f(x) = 3 - 5x$ find the average gradient between
   a) $x = 3$ and $x = 1$
   b) $x = -2$ and $x = 0$
   c) $x = 10$ and $x = 12$

   Why are these answers the same?
4. $s(t) = 80t - 4t^2$ is the distance in metres when a stone is thrown in the air. t is time in seconds.
   a) How high is the stone after 2 seconds?
   b) How long is the stone in the air?
   c) What is the average speed between 3 and 5 seconds?

Activity 2

1. Differentiate from first principles:
   a) $f(x) = 4$
   b) $f(x) = 1 - x^2$
   c) $f(x) = 3x - 1$
   d) $f(x) = -x^3$
   e) $y = \frac{3}{x}$
2. Find the gradient of the tangent to each of the following functions at the given points by using first principles:
   a) $y = 2x + 3$ at $x = 1$
   b) $f(x) = x^2 + x$ at $x = -2$
   c) $f(x) = -1$ at $x = 7$
   d) $f(x) = \frac{1}{x}$ at $x = 4$
   e) $y = \frac{2}{x^2}$ at $x = 1$
   f) $f(x) = \sqrt{x}$ at $x = 9$