Lesson 23

CALCULUS (6)

Cubic graphs

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and Algebra
Assessment standard 12.2.7(a)
Investigate and use instantaneous rate of change of a variable when interpreting models of situations:

- demonstrating an intuitive understanding of the limit concept in the context of approximating the rate of change or gradient of a function at a point
- establishing the derivative of the following functions from first principles
  $f(x) = b;\ f(x) = x^2;\ f(x) = \frac{1}{x};\ f(x) = x;\ f(x) = x^3$
  and then generalise to the derivative of $f(x) = x^n$.

Overview

In this lesson you will:

- find equations of cubic functions
- solve problems when you have been given a graph
- understand local maximums and minimums

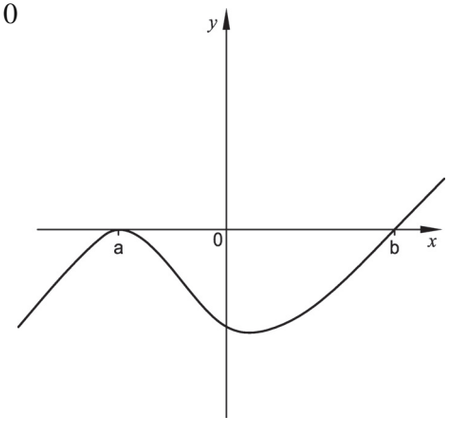
LC DVD

Lesson

What we have learnt about cubic functions

$$y = ax^3 + bx^2 + cx + d$$

1. If $a > 0$ and there are two stationary points.
   Shape: 
2. If $a < 0$ and there are two stationary points.
   Shape: 
3. If $a > 0$ and there is one stationary point. (It is a stationary point)
   Shape: 
4. If a < 0 and there is one stationary point. (It is a stationary point)
   Shape: 
5. If the graph turns at the x-axis there will be two x-intercepts at that point.
   $(x - a)^2(x - b) = 0$



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Example 1

Data was analysed at a weather station. It was found that the humidity H in relation to the temperature t (°C) could be expressed by means of the formula $H = \frac{1}{10}(-t^3 + 24t^2 - 84t + 80)$

a) Calculate the humidity at 0° C.

b) Calculate the temperature(s) when the humidity is 0.

c) At what temperature does the humidity reach a maximum in the interval [0°; 20°]? Calculate the maximum.

d) Draw the sketch graph of H in relation to t in the interval [0°; 20°].

Solution

a) $H = \frac{1}{10}(80)$

b) $0 = -t^3 + 24t^2 - 84t + 80$

$t^3 - 24t + 84t - 80 = 0$

$f(t) = t^3 - 24t^2 + 84t - 80$

$f(t) = (t - 2)(t^2 - 22t + 40)$

$f(t) = (t - 2)(t - 2)(t - 20)$

$(t - 2)^2(t - 20) = 0$

$t = 2$ or $t = 20$

c) $H = \frac{1}{10}t^3 + \frac{24}{10}t^2 - \frac{84}{10}t + 8$

$\frac{dH}{dt} = -\frac{3}{10}t^2 + 48/10t - \frac{84}{10}$

$0 = -3t^2 + 48t - 84$

$0 = 3t^2 - 48t + 84$

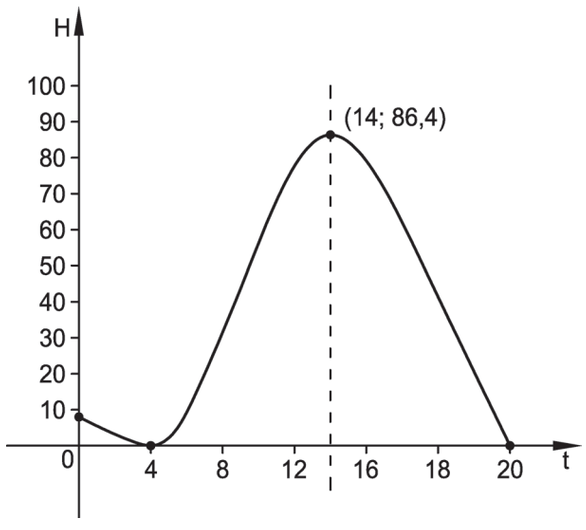
$0 = (t - 2)(3t - 42)$

$t = 2$ or $t = \frac{42}{3} = 14$

Min Max

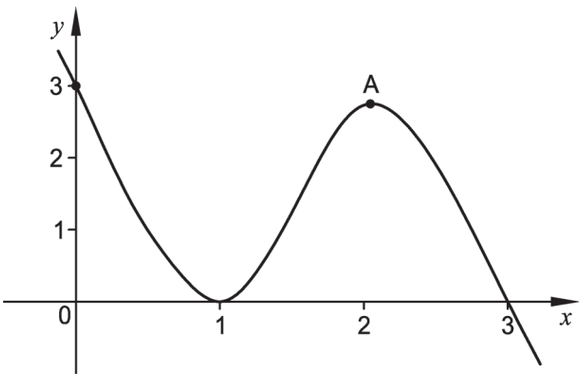
Maximum $\frac{432}{5} = 86{,}4$°C

d)



Example 2

Look at this graph and find the equation if $y = ax^3 + bx^2 + cx + d$



Solution

Step 1: Put in the x-intercepts

$y = a(x - 1)^2(x - 3)$

Step 2: Substitute the point (0; 3) to find a.

$3 = a(-1)^2(-3)$

$3 = -3a$

$a = -1$

Equation: $y = -1(x - 1)^2(x - 3)$

Now you solve the next two questions based on the previous work.

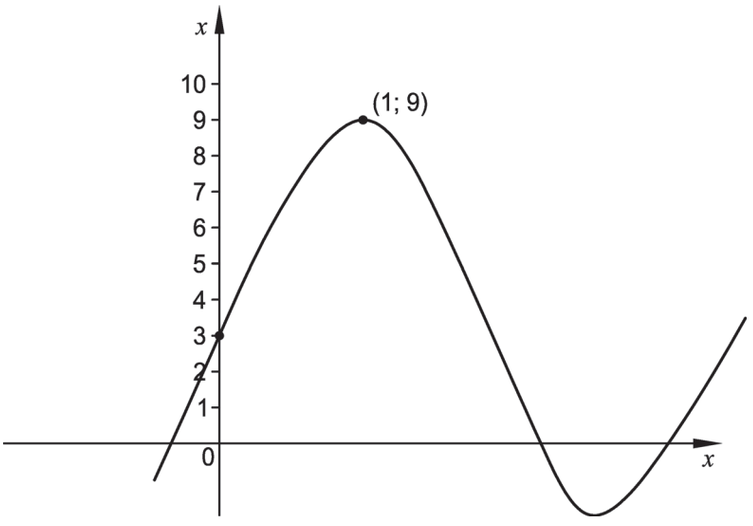
1. Find the co-ordinates of the turning point A.
2. Find the equation of the tangent at $x = -1$

Solutions:

1. $\frac{7}{3}; \frac{32}{27}$
2. $y = -20x - 4$

Example 3

Look at this graph and find the equation if $y = x^3 + ax^2 + bx + c$



Solution

We know c because it is the y-intercept, so $y = x^3 + ax^2 + bx + 3$

To get a and b we need two further bits of information.

1. The point (1; 9) is on the graph so

   $9 = 1 + a + b + 3$

   $5 = a + b$

2. The gradient is 0 when $x = 1$ (this point is also a turning point)

   $\frac{dy}{dx} = 3x^2 + 2ax + b$

   $0 = 3 + 2a + b$

   $-3 = 2a + b$

Now solve the above two equations simultaneously

$$\begin{array}{lll} 2a + b = -3 & & ② \\ a + b = 5 & \text{subtract} & ① \\ \hline a = -8 & & \end{array}$$

Substitute into ① to find b

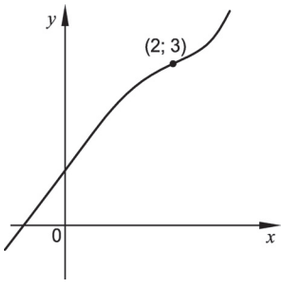
$-8 + b = 5$

$b = 13$

Equation: $y = x^3 + 8x^2 + 13x + 3$

Example 4

This is the graph of a cubic function with a point of reflection at point (2; 3). Draw a sketch graph of the derivative.



Solution

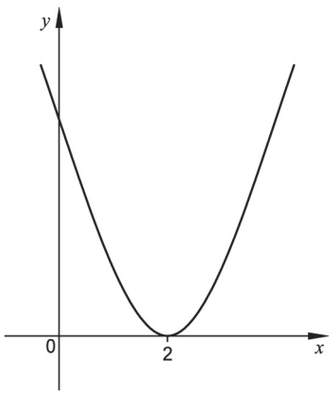
The derivative is the gradient

The gradient is zero when $x = 2$

Otherwise the gradient is always positive

The derivative is the graph of a parabola (positive) above the x-axis but zero (x-intercept) at $x = 2$.

Now you find the equation of the derivative if the gradient of the function is 8 when $x = 0$



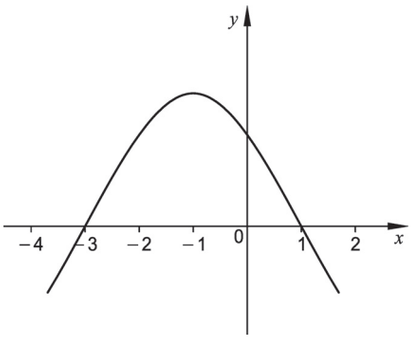
Solution

$y = 2x^2 - 8x + 8$

Example 5

This is the graph of $f'(x)$.

Draw a rough sketch of $f(x)$ if $f(-3) = -2$



Solution

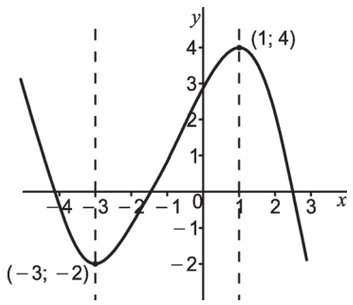
- This is the derivative so $f(x)$ is a cubic function.
- The co-efficient of x^2 is negative so the co-efficient of x^3 must be negative.
- The gradient is zero at x = −3 and x = 1 so $f(x)$ turns at $x = -3$ and $x = 1$.
- Shape 

Now try this

Find the values of x if x. $f'(x) > 0$

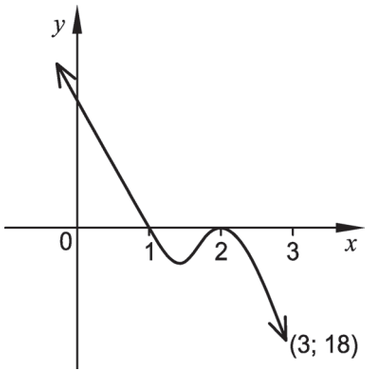
Solution

$x < -3$ or $0 < x < 1$



Activity

1. The curve shown intersects the x-axis at (1; 0), passes through (3; −18) and has a turning point at (2; 0). Determine the equation defining the curve.



2. The equation of a curve is $y = -x^3 + bx^2 + cx$ where b and c are constant, real numbers. The gradient of the curve at the point (1; 0) is 0. Calculate the values of b and c.

3. Use the following information to sketch the graph of $f : x \to ax^3 + bx^2 + cx + d$
(Show all important turning points, e.g. intercepts and turning points.)

$f(0) = 2$ $f(-4) = 0$

$f'(-2) = f'(1) = 0$

$f(x) \leq 5$ if $x < 0$

$f(x) \geq 1$ if $x > 0$

(No calculations are necessary.)

4. The function $y = f(x)$ is of the third degree. The parabola in the sketch represents the curve of $y = f'(x)$

   a) What is the gradient of the tangent to the curve of $y = f(x)$ if $x = 0$?

   b) For which values of x will there be a tangent to the curve of $y = f(x)$ which will be parallel to the tangent in (a).

   c) For which values of x will $f(x)$ be decreasing.

   d) Write down the x co-ordinates of the turning points of the curve of f and state whether these points will be local maxima or minima.

   e) If it is further given that the solution $f(x) = 0$ is $x = 0$ or $x = 3$ or $x = 7$ use the information at your disposal and sketch a graph of f.

5. The graphs of $f : x \to x^2 - x^3$ and $g : x \to x - ax^2$ have the same gradient at $x = \frac{1}{3}$. Calculate

   a) The value of a.

   b) Another value of x for which the two graphs will have equal gradients.