Lesson 25

CALCULUS (8)

Rate of change

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and Algebra
Assessment standard 12.2.7(e)
Solve practical problems involving optimisation and rates of change.

Overview

In this lesson you will

- discuss displacement, velocity and acceleration and solve practical problems involving these concepts.
- look at other rates of change and solve practical problems.

Lesson

LC DVD

From science

$s = ut + \frac{1}{2}at^2$ Where s is displacement, and u and a are constants

$\frac{ds}{dt} = u + at$ This is velocity (instantaneous speed)

$\therefore V = \frac{ds}{dt}$

$V = u + at$

$\therefore \frac{dv}{dt} = a$ Which is acceleration

Example 1

This displacement of a body is given by the formula $s = 30t - 2t^2$ where s is distance in metres and t is time in seconds?

a) What is the height of the body after 1 second?
b) After how many seconds will the body be at a height of 100 m?
c) What is the velocity of the body after 1 second?
d) After how many seconds will the body reach its maximum height, and what is the maximum height?
e) What is the initial velocity of the body?
f) How long is the body in the air?
g) At what speed does the body hit the ground?
h) What is the acceleration of the body?

Solutions

a) $s = 30(1) - 2(1)$

$= 28$ metres

b) $30t - 2t^2 = 100$

$0 = 2t^2 - 30t + 100$

$0 = t^2 - 15t + 50$

$0 = (t - 5)(t - 10)$

$t = 5$ or $t = 10$

c) $V = 30 - 4t$

$V = 30 - 4$

$= 26 \text{ m·s}^{-1}$

d) $V = 0$

$30 - 4t = 0$

$30 = 4t$

$t = 7\frac{1}{2}$

Now substitute to find the height

$s = 30t - 2t^2$

$s = 30(7{,}5) - 2(7{,}5)^2$

$s = 337{,}5$ metres

e) Velocity when $t = 0$

$V = 30 \text{ m·s}^{-1}$

f) Make displacement equal zero.

$0 = 30t - 2t^2$

$0 = 2t(15 - t)$

$t = 15$

15 seconds

g) Velocity at 15 seconds

$V = 30 - 4t$

$V = 30 - 4(15)$

$= -30$

at 30 m·s^{-1}

h) $\frac{dV}{dt} = -4$

4 m.s^{-2}

Example 2

A particle moves so that its velocity at any time (t) in seconds is given by:

$V = 2t + 3$

a) Find an expression for the displacement (s) in terms of t.

b) If the displacement is 6 metres after one second, what will the displacement be after 5 seconds?

c) What is the acceleration of the particle?

Solution:

a) $v(t) = 2t + 3$

Remember $v(t) = s'(t)$, and we used the power rule to differentiate with. So we reverse that, by undoing what we did when we differentiated.

Remember: If $f(x) = ax^n \rightarrow f'(x) = n \cdot a \cdot x^{n-1}$

So if $f'(x) = ax^n \rightarrow f(x) = \frac{ax^{n+1}}{n+1}$

Something else to remember is that the derivative of a constant is zero. So we need to make provision for this.

So: $s(t) = \frac{2t^{1+1}}{2} + 3t + c$
$= t^2 + 3t + c$ } We add one to the power and divide by the new power.

b) $s(1) = 6$: $(1)^2 + 3(1) + c = 6$

$\therefore\ c = 6 - 4 = 2$

$\therefore\ s(t) = t^2 + 3t + 2$

and $s(5) = 5^2 + 3(5) + 2 = 25 + 15 + 2$

$= 42$ metres

c) $a(t) = v'(t) = 2$ m/s^2

Example 3

A stone is thrown vertically upwards off the roof of a building and its displacement after t seconds is given by $s = 20t - 5t^2$. t is time in seconds and s is distance in metres.

a) What is the stone's initial velocity?

b) What is the maximum height above the building reached by the stone?

c) If the stone is thrown so that it misses the building on its way down, and takes a total time of 6 seconds to reach the ground, how high is the building?

Solution:

a) $V = \frac{ds}{dt}$

$V = 20 - 10t$

Initial velocity $t = 0$: $v = 20 - 10(0) = 20$ m/s

b) $V = 0$

$20 - 10t = 0$

$t = 2$ seconds

Reaches maximum height after 2 seconds

$s = 20(2) - 5(4)$

$= 20$ m

c) $s = 20t - 5t^2 + c$
where c will be this extra distance that the stone will travel

Displacement is 0

when $t = 6$

$0 = 120 - 180 + c$

$c = 60$ m

Building is 60 m high

Rate means the derivative gradient

Increase means gradient is positive

Decrease means gradient is negative

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Example 4

The manner in which the temperature, T° C, at the centre of a smelting pot in a blast furnace increases with time is given by $T = t^2(45 - t) \times 10^{-1} + 15$ where t is time in minutes.

a) Find the rate of increase in temperature when $t = 5$ minutes

b) After how many minutes will the temperature be increasing at a rate of 64,8° C per minute.

Solution:

a) $T = \frac{t^2}{10}(45 - t) + 15$

$T = \frac{9t^2}{2} - \frac{1}{10}t^3 + 15$

$\frac{dT}{dt} = 9t - \frac{3}{10}t^2$

After 5 minutes rate of increase is $9(5) - \frac{3}{10}(25) = 37{,}5$° C per minute

b) $\frac{dT}{dt} = 64{,}8$

$9t - \frac{3}{10}t^2 = \frac{648}{10}$

$90t - 3t^2 = 648$

$3t^2 - 90t + 648 = 0$

$t^2 - 30t + 216 = 0$

$(t - 18)(t - 12) = 0$

$t = 18$ or $t = 12$

At 18 minutes and at 12 minutes

Example 5

Water is leaking from a tank and the depth of the water in the tank (y mm) after t minutes is given by $y = 16 - \frac{1}{8}t - \frac{1}{4}t^3$.

a) At what rate is the depth of the water decreasing when $t = 2$?

b) After how many minutes will the depth of the water be decreasing at a rate of $6\frac{7}{8}$ m per minutes?

Solution:

a) Rate $\frac{dy}{dt}$

$\frac{dy}{dt} = -\frac{1}{8} - \frac{3}{4}t^2$

$= -3\,\frac{1}{8}$

At 2 minutes

Decreasing at a rate of $-\frac{3}{8}$ mm per minute

b) $\frac{dy}{dt} = -\frac{55}{8}$

$-\frac{1}{8} - \frac{3}{4}t^2 = -\frac{55}{8}$

$\frac{54}{8} = \frac{3}{4}t^2$

$6t^2 = 54$

$t^2 = 9$

$t = 3$

3 minutes

Activity

1. A particle moves in such a way that distance(s) in metres after t seconds will be given by $s = 8t^2 - t^3$. Find its position, velocity and acceleration after 3 seconds.
2. A ball is thrown vertically upwards. Its height (h metres) at t seconds is given by $s = 36t - 4t^2$
   a) Find at what height the ball is after 2 seconds.
   b) After how many seconds will the ball again hit the ground?
   c) Calculate the velocity of the ball when it hits the ground.
   d) After how many seconds will the ball reach its maximum height?
   e) Calculate the maximum height of the ball.
   f) What is the acceleration of the ball?
3. If a particle moves so that $V = t^2 - t + 2$
   a) Find an expression for s
   b) Find s at $t = 6$ given that $s = 3$ and $t = 0$
   c) Find an expression for the acceleration of the particle.
4. A motorcyclist takes off from a stop light and stops when he reaches a second one. His distance, $f(t)$ in metres, from the 1st light t seconds after he takes off is given by the function $f(t) = 6t^2 - 0{,}4t^3$. Determine
   a) How fast he was travelling when $t = 5$?
   b) The time it took him to reach the 2nd light.
   c) The distance between the two stop lights.
5. The volume of water in a tank is given by the equation $V = 5 + 10t - t^2$ (V is volume in m^3, t = time in minutes).
   a) Find the rate at which the volume is increasing when $t = 2$ minutes.
   b) At what time does the volume start decreasing?
   c) What is the maximum volume?
6. The sweetness, s, of a peach, t days after it began to ripen is given by $s = 3\,300 + (t - 15)^3$
   a) At what rate is s increasing when $t = 0$?
   b) The peach falls off the tree (ie it is fully ripe) when s stops increasing. How many days does the peach stay on the tree?
7. The volume V(cm^3) of water remaining in a leaking pail after t seconds is $V = 2\,000 - 40t + 0{,}2t^2$
   a) Find the average rate of change of volume from $t = 30$ to $t = 40$.
   b) How fast is the volume decreasing when $t = 30$?
8. A water tank with an inlet and an outlet is used to water a garden. The equation $D = 3 + \frac{1}{2}t^2 - \frac{1}{4}t^3$ gives the depth of water in metres where t is the time in hours that has elapsed since 09:00.
   a) What is the depth of the water at 11:00?
   b) At what rate does the depth of the water change at 12:00?
   c) At what time will the water be decreasing at a rate of 8 metres per hour?
   d) At what time will the inflow of water be the same as the outflow of water?

ANSWERS AND ASSESSMENT

Lesson 18

Activity 1

1. a) -1 b) 4
2. a) -6 b) -6
3. a) -5 b) -5 c) -5
   It is a straight line.
4. a) 144 m b) 20 seconds c) 48 $m \cdot s^{-1}$

Activity 2

1. a) 0 b) $-2x$ c) 3 d) $-3x^2$ e) $-\frac{3}{x^2}$
2. a) 2 b) -3 c) 0 d) $-\frac{1}{16}$ e) -4
   f) $\frac{1}{16}$

Lesson 19

1. $\lim_{x \to 3} (2x^2 + 4 - 3x) = 2(3)^2 + 4 - 3(3) = 18 + 4 - 9 = 13$
2. $\lim_{x \to 1} \frac{x^2 - 3x}{2x + 5} = \frac{1^2 - 3(1)}{2(1) + 5} = \frac{-2}{7}$
3. -1
4. $\lim_{x \to \sqrt{2}} \frac{x - 3}{2x + 5} = \frac{\sqrt{2} - 3}{2\sqrt{2} + 5}$
5. $\lim_{x \to 0} (2x^2 + 4 - 3x) = 4$
6. -6
7. 12
8. no limit
9. 2
10. $\frac{1}{12}$
11. $\frac{13}{2}$
12. 4
13 0
14. $\lim_{x \to \frac{3}{4}} \frac{16x^2 - 9}{3 - 4x} = \lim_{x \to \frac{3}{4}} \frac{(4x - 3)(4x + 3)}{3 - 4x} = \lim_{x \to \frac{3}{4}} -(4x + 3) = -(3 + 3) = -6$
15. $\lim_{x \to 8} \frac{x^{\frac{2}{3}} + 3\sqrt{x}}{4 - \frac{16}{x}} = \frac{(2^3)^{\frac{2}{3}} + 3(2^3)^{\frac{1}{2}}}{4 - 2} = \frac{4 + 6\sqrt{2}}{2} = 2 + 3\sqrt{2}$
16. 27
17. no limit
18. $-\frac{1}{4}$
19. no limit
20. no limit
21. a) does not exist b) 1 c) 1 d) no limit
    e) no limit f) 4 g) 2

Lesson 20

1. a) 0 b) 0
   c) 0 d) $-3x^{-4}$
   e) $-x^{-2}$ f) $\frac{1}{4}$
   g) $-18x^{-7}$ h) $f(x) = x^{\frac{1}{2}}$
   $\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
   i) $f(x) = \frac{1}{2}x^{\frac{3}{4}}$
   $\therefore f'(x) = \frac{3}{8}x^{-\frac{1}{4}}$
   j) $f(x) = x^{-1} + 2x^{-1}$
   $f'(x) = -x^{-2} - 2x^{-2} = -3x^{-2}$
2. $s = 5t^{-1} - 4t$
   $\frac{ds}{dt} = -5t^{-2} - 4$
3. $m = \frac{n^4 - 2n^2 + 1}{n^2}$
   $m = n^2 - 2 + n^{-2}$
   $\frac{dm}{dn} = 2n - 2n^{-3}$
4. $f(x) = 9x^4 - 6x^3 + x^2$
   $f'(x) = 36x^3 - 18x^2 + 2x$
5. $\frac{d}{dx}(x^2 - x + x^{-1})$

$= 2x - 1 - x^{-2}$

6. $s'(t) = 0{,}2t^{-0{,}8}$

7. $y = x + 2 + x^{-1}$

$\frac{dy}{dx} = 1 - x^{-2}$

8. $\frac{d}{dt}[t^{\pi}] = \pi t^{\pi - 1}$

9. $s = \frac{2}{3}t\frac{1}{2} + \frac{1}{3}t^{-\frac{1}{2}}$

$\frac{ds}{dt} = \frac{1}{3}t^{-\frac{1}{2}} - \frac{1}{6}t^{-\frac{3}{2}}$

10. $f(x) = \frac{(x-1)(x^2 + 4x - 3)}{(x-1)}$

$f'(x) = 2x + 4$

11. $D_x\left[\frac{2\sqrt{x} - x^2 + 6}{\sqrt[3]{5x^2}}\right] = D_x\left(\frac{2x^{\frac{1}{2}-\frac{2}{3}}}{\sqrt{5}} - \frac{x^{2-\frac{2}{3}}}{\sqrt{5}} + \frac{x^{-\frac{2}{3}}}{\sqrt{5}}\right)$

$= \frac{1}{\sqrt{5}} D_x\left(2x^{-\frac{1}{6}} - x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right)$

$= \frac{1}{\sqrt{5}}\left(-\frac{1}{3}x^{-\frac{7}{6}} - \frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{5}{3}}\right)$

$= -\frac{1}{3x\sqrt{5x}} - \frac{4\sqrt[3]{x}}{3\sqrt{5}} - \frac{2}{3\sqrt{5}x\sqrt[3]{x^2}}$

12. $\frac{d}{du}[1 + u^{-2} - u^2 - 1]$

$= -2u^{-3} - 2u$ 13. $\frac{dy}{dx} = -6x$ Gradient 6

14. $-3 - 4x = 13$

$-4x = 16$

$\therefore x = -4$

$(-4; -18)$

15. $\frac{dy}{dx} = 0$

$\therefore x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2$ or $x = 1$

$\left(-2; \frac{10}{3}\right)\left(1; -\frac{11}{6}\right)$ 16. $2x - 4 = -6$

$\therefore 2x = -2$

$x = -1$

$(-1; 7)$

17. The gradient of the tangent at $x = 2$ is 3.

18. $x^2 - 2x - 1 = 2$

$\therefore x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

19. $f'(x) = 4$ when $x = 1$

$f'(x) = 3x^2 + 2x + a$

$\therefore 5 + a = 4$

$a = -1$

$(1; -2)$ is on the curve

$y = -2 \quad x = 1$

$-2 = 2 + a - b$

$b = 2$

20. $d'(t) = 3t^2$

$300\ \text{m}\cdot\text{s}^{-1}$

21. $V(t) = t^{\frac{3}{2}}$

$V'(t) = \frac{3}{2}t^{\frac{1}{2}}$

$V'(t) = \frac{3}{2}\sqrt{9}$

$= \frac{9}{4}$

$\frac{9}{4}$ units per year

Lesson 21

Activity 1

1. Since the tangent is parallel to the line $y = 11x$, we have that $m_T = 11$

   Thus: $f'(x) = 3x^2 + 2x + 3$

   But $m_T = f'(x_T)$

   $\therefore\ 11 = 3x^2 + 2x + 3$

   $\therefore\ 3x^2 + 2x + 3 - 11 = 0$

   $\therefore\ 3x^2 + 2x - 8 = 0$

   $\therefore\ (3x - 4)(x + 2) = 0$

   $\therefore\ x = \frac{4}{3}$ or $x = -2$

   For $x_T = \frac{4}{3}$: $y_T = \left(\frac{4}{3}\right)^3 + \left(\frac{4}{3}\right)^2 + 3\left(\frac{4}{3}\right) + 5$

   $= \frac{64}{27} + \frac{16}{9} + 9$

   $= \frac{64 + 48 + 243}{27}$ $\quad \therefore\ T_1\left(\frac{4}{3}; \frac{355}{27}\right)$

   $= \frac{355}{27}$

   For $x_T = -2$: $y_T = (-2)^3 + (-2)^2 + 3(-2) + 5 = -8 + 4 - 6 + 5 = -5$

   $\therefore\ T_2\ (-2\ ; -5)$

   Eq tangent 1: $y - y_T = m_T(x - x_T)$

   $\therefore y - \frac{355}{27} = 11\left(x - \frac{4}{3}\right)$

   $\therefore\ 27y - 355 = \left(11x - \frac{44}{3}\right)27$

   $\therefore\ 81y - 1\ 065 = 891x - 1\ 188$

   $\therefore\ 81y = 891x - 123$

   $\therefore\ y = 11x + 1{,}52$

   Eq tangent 2: $y - y_T = m_T(x - x_T)$

   $\therefore\ y - (-5) = 11(x - (-2))$

   $\therefore\ y + 5 = 11x + 22$

   $\therefore\ y = 11x + 17$

2. x_T is given as 2:

   For m_T: $f'(x) = 2x - 3$ For y_T: $y_T = f(2)$

   $\therefore f'(2) = 4 - 3 = 1$ $\quad = (2)2 - 3(2) + 7$

   $\therefore\ m_T = 1$ $\quad = 4 - 6 + 7$

   $\therefore y_T = 5$

   Eq tgt: $y - f(x_T) = f'(x_T)(x - x_T)$

   $\therefore\ y - 5 = 1(x - 2)$

   $\therefore\ y - 5 = x - 2$

   $\therefore\ y = x + 3.$

3. Find the value of x where they cut. Then show that the product of the gradients at that point is -1.

   First find where the curves intersect

   $x^2 = -x^2 + \frac{1}{2}$

   $2x^2 = \frac{1}{2}$

   $4x^2 = 1$

$x = \pm\frac{1}{2}$

For $y = x^2$ For $y = -x^2 + \frac{1}{2}$

$\frac{dy}{dx} = 2x$ $\frac{dy}{dx} = -2x$

at $x = \frac{1}{2}$ $m_1 = 1$ at $x = \frac{1}{2}$ $m_2 = -1$

at $x = -\frac{1}{2}$ $m_1 = -1$at $x = -\frac{1}{2}$ $m_2 = 1$

$\therefore m_1 \times m_2 = -1$

At both points the curves intersect at right angles.

We say the curves intersect orthogonally.

4. a) $y - \frac{6}{x}$

$y = 6x^{-1}$

$\frac{dy}{dx} = \frac{-6}{x^2}$

Slope of tangent at $x = -2$ is $-\frac{6}{4} = -\frac{3}{2}$

Desired point is $(-2; -3)$

Equation of normal $y + 3 = \frac{2}{3}(x + 2)$

$3y + 9 = 2x + 4$

$3y = 2x - 5$

b) $\therefore 6x = \frac{2}{3}x - \frac{5}{3}$

$18 = 2x^2 - 5x$

$2x^2 - 5x - 18 = 0$

$(2x - 9)(x + 2) = 0$

$y = \frac{6}{x}$ and $y = \frac{2}{3}x - \frac{5}{3}$ intersect

The normal cuts the curve again at $\left(-\frac{9}{2}; -\frac{4}{3}\right)$

5. $h(x) = -3x^3 + 2x^2 - 3x + 5$ and $x_T = -1$

Thus: For $m_T \Rightarrow h'(x) = -9x^2 + 4x - 3$

$\therefore m_T = h'(-1)$

$\therefore m_T = -9(-1)2 + 4(-1) - 3$

$\therefore m_T = -9 - 4 - 3$

$\therefore m_T = -16$

and $m_N = \frac{1}{16}$ since tangent is perpendicular to the normal at the point of tangency

For $y_T \Rightarrow y_T = h(x_T)$

$\therefore y_T = -3(-1)^3 + 2(-1)^2 - 3(-1) + 5$

$\therefore y_T = 3 + 2 + 3 + 5$

$\therefore y_T = 13$

Thus Eq tgt: $y - y_T = m_T(x - x_T)$

Eq norm: $y - y\mathrm{T} = \mathrm{mN}\ (x - x\mathrm{T})$

$\therefore y - 13 = -16(x - (-1))$

$\therefore y - 13 = \frac{1}{16}(x - (-1))$

$\therefore y - 13 = -16x - 16$

$\therefore 16y - 208 = x + 1$

$\therefore y = -16x - 3$

$\therefore 16y = x + 209$

$\therefore y = \frac{x}{16} + \frac{209}{16}$

To find where the tangent cuts the curve again:

$h(x) = -3x^3 + 2x^2 - 3x + 5$ … (1)

and $y = -16x - 3$ … (2)

(1) $\rightarrow$ (2) : $-16x - 3 = -3x^3 + 2x^2 - 3x + 5$

$\therefore 3x^3 - 2x^2 - 13x - 8 = 0$

The tangent touches the cubic and we therefore have a double root at $x = -1$.

Thus $(x + 1)^2$ is a factor of this new equation :

$3x^3 - 2x^2 - 13x - 8 = (x^2 + 2x + 1)(3x - 8)$

Thus the tangent cuts the curve again at $x = \frac{8}{3}$

and $f\left(\frac{8}{3}\right) = -3\left(\frac{8}{3}\right)^3 + 2\left(\frac{8}{3}\right)^2 - 3\left(\frac{8}{3}\right) + 5$

$= -\frac{512}{9} + \frac{128}{9} - 8 + 5$

$= \frac{-512 + 128 - 72 + 45}{9}$

$= -\frac{411}{9}$

Cuts again at point $\left(\frac{8}{3}; -\frac{411}{9}\right)$

Activity 2

1. $y + 6x + 1 = 0$
2. $4y + 4x + 9 = 0$
3. $y = 2x + 4$
4. $9y = -6x + 11$
5. $(1; 2)$ or $(-1; -2)$

Lesson 22

Activity 1

1. $\frac{dy}{dx} = 2x - 6$

   increasing decreasing stationary point

   $2x - 6 \geq 0$ $x \leq 3$ $(3; -7)$

   $x \geq 3$
2. $\frac{dy}{dx} = -4$ Always decreasing, no stationary points
3. $f'(x) = 3x^2 + 6x$

   $3x^2 + 6x$

   $3x(x + 2)$

   increasing decreasing stationary point

   $x \leq -2$ and $x \geq 0$ $-2 \leq x \leq 0$ $(-2; 22)$ $(0; 2)$
4. $p'(x) = 3x^2 - 12$

   $3(x^2 - 4)$

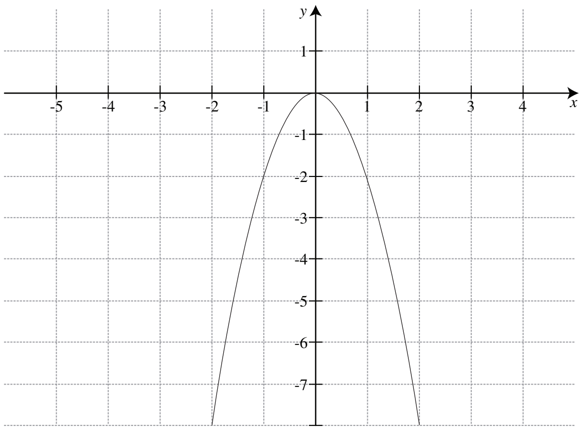
   $= 3(x - 2)(x + 2)$

   increasing decreasing stationary point

   $x \leq -2$ and $x \geq 2$ $-2 \leq x \leq 2$ $(-2; 17)$ $(2; -3)$
5. $\frac{dy}{dx} = 5x^4$

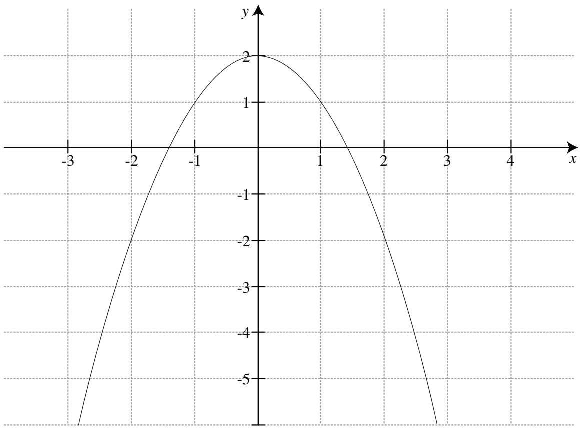
   increasing everywhere $(-\infty; \infty)$ stationary point $(0; 0)$

6 (a) $y = -2x^2$; $[-5;9]$



The graph is increasing from [–5;0) and decreasing from (0;9]. At $x = 0$ it is stationary.
Mathematically: $\frac{dy}{dx} = -4x$ and $-4x > 0$ *iff* $x < 0$. So if $x < 0$ it will increase. It will then decrease on the $x > 0$ side.

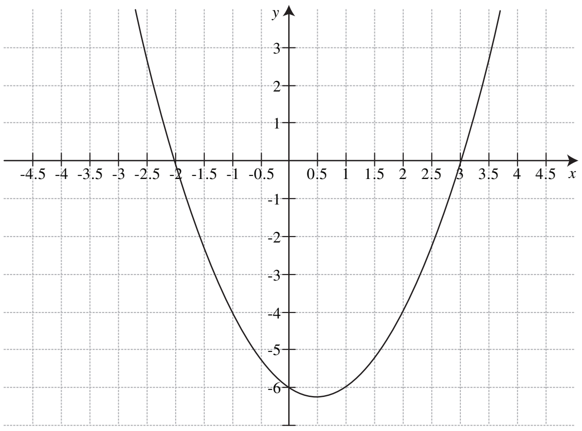
(b) $y = 2 - x^2$; [−3;6]



The graph is increasing from [–3; 0) and decreasing from (0; 6] whilst it is stationary at $x = 0$.

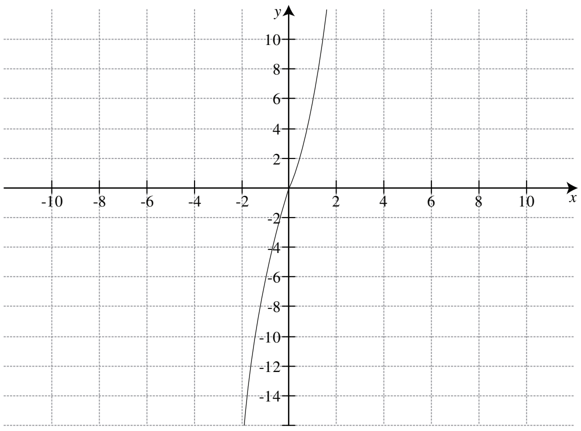
Mathematically: $\frac{dy}{dx} = -2x$ and $-4x > 0$ *iff* $x < 0$. So if $x < 0$ it will increase. It will then decrease on the $x > 0$ side.

(c) $y = x^2 - x - 6$; [−4; 4]



The graph decreases on [–4; 0) and increases on (0;4]. It is stationary at $x = \frac{1}{2}$.
Mathematically: $\frac{dy}{dx} = 2x - 1$ and $2x - 1 > 0$ *iff* $x > 2$. So if $x < \frac{1}{2}$ it will decrease.
It will then increase on the $x > \frac{1}{2}$ side.

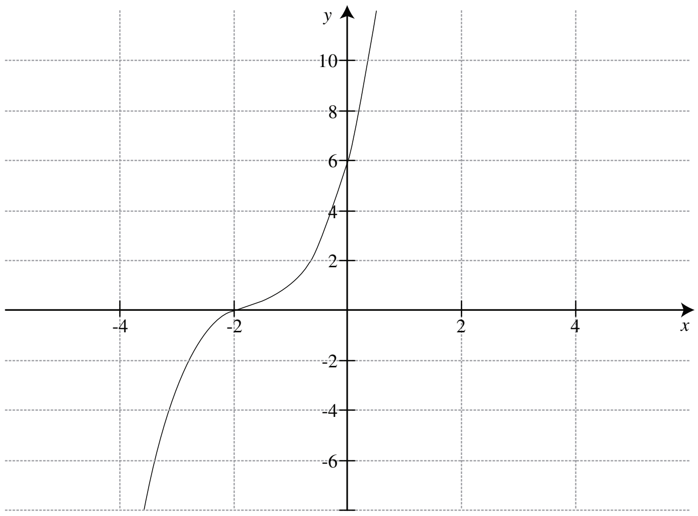
(d) $y = x^3 + 5x$; [−7;9]



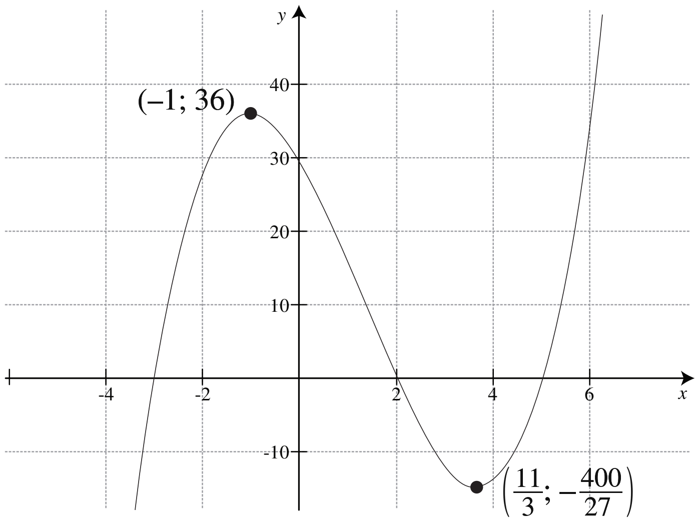
It is increasing over its whole domain with no stationary points.
Mathematically: $\frac{dy}{dx} = 3x^2 + 5$ and $3x^2 + 5 > 0 \ \forall \ x \in \mathbb{R}$. So is an increasing function.

Activity 2

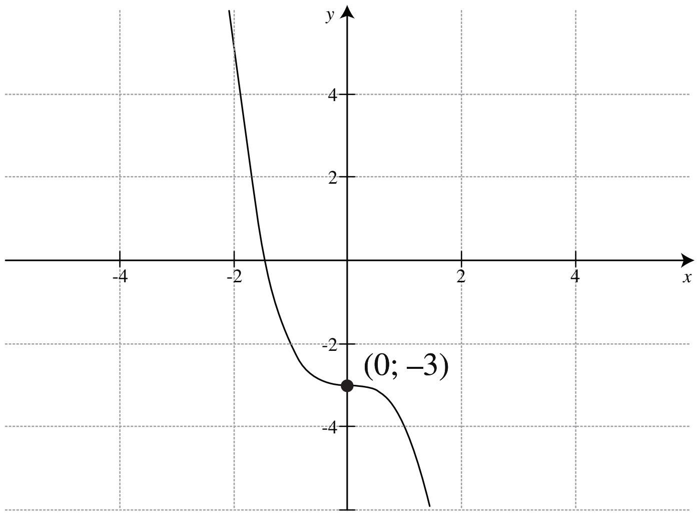
1.



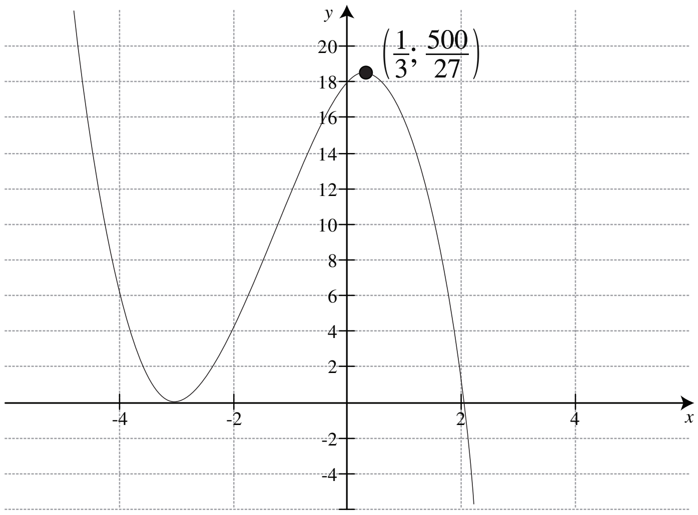
2.



3.



4.



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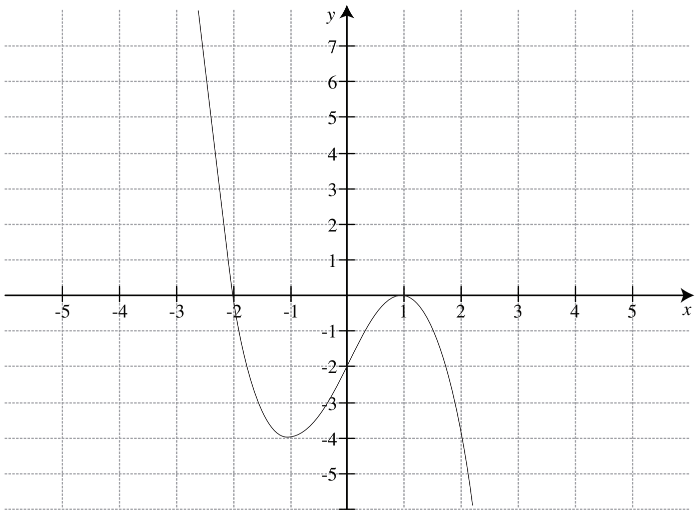

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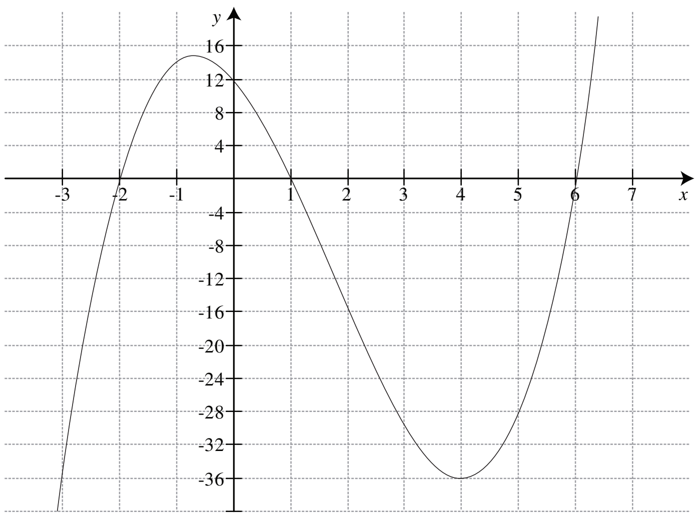

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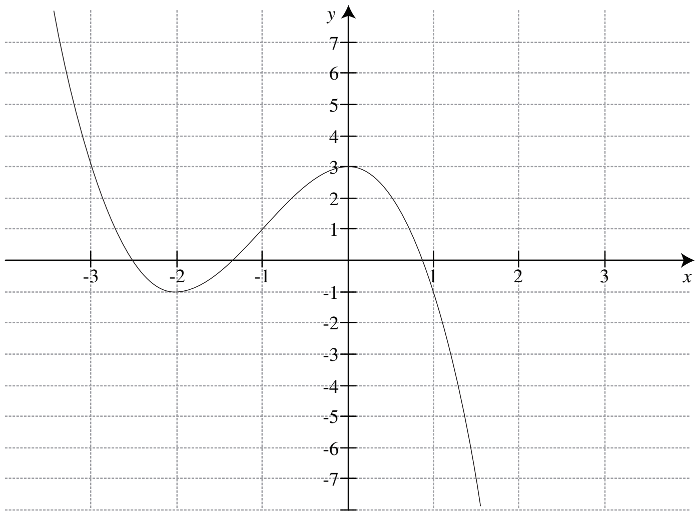
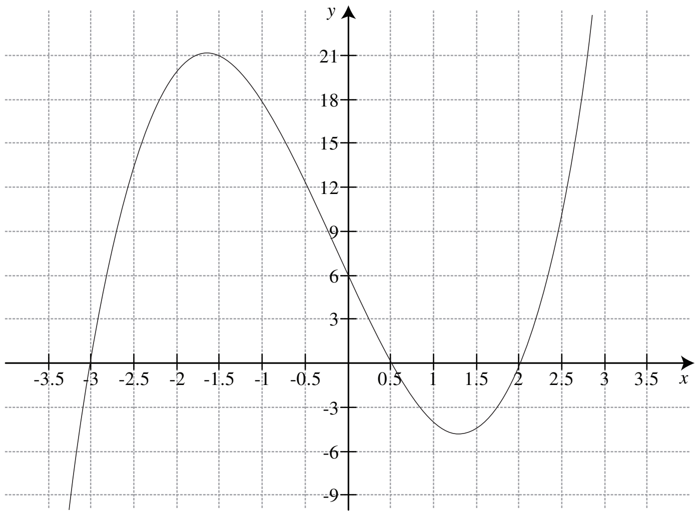


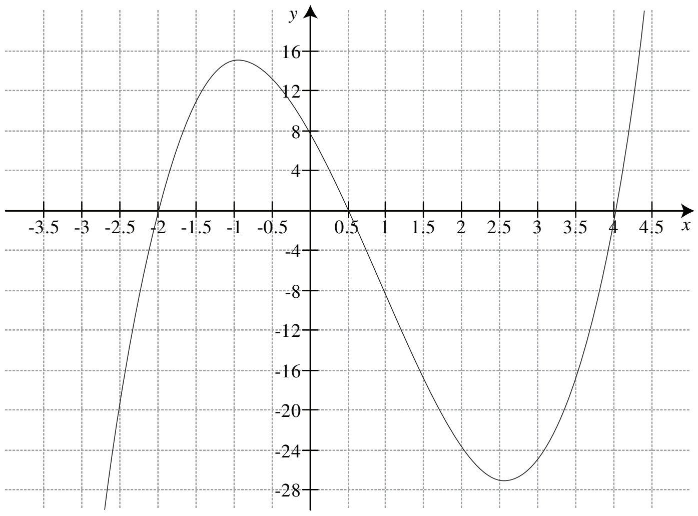
y
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-3
-2
-1
1
2
3
x

8.



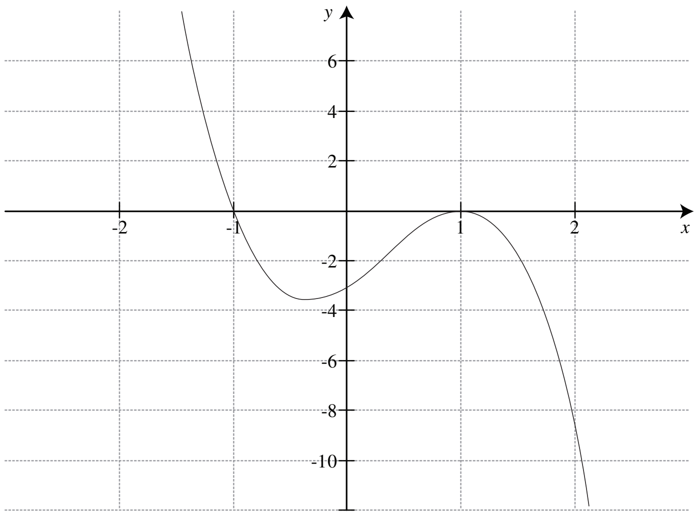
y
21
18
15
12
9
6
3
-3
-6
-9
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
3
3.5
x

9.

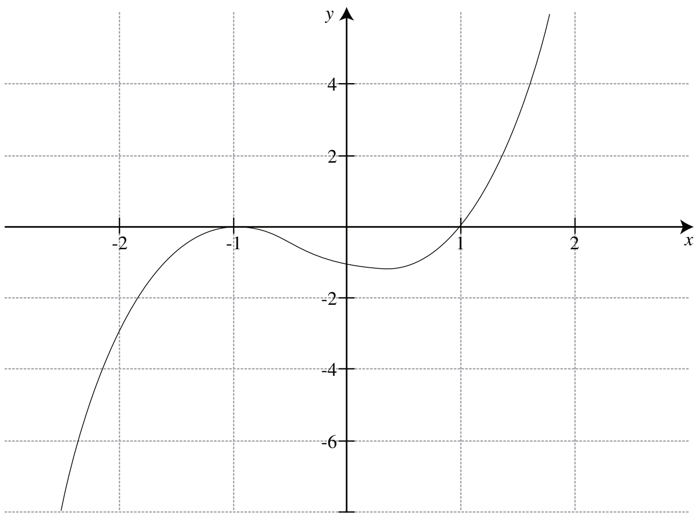


y
16
12
8
4
-4
-8
-12
-16
-20
-24
-28
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
3
3.5
4
4.5
x

10.



11.



Lesson 23

1. a) i) The zeros of f are 0; 6 and 6 (the x-axis is a tangent to the curve).

$\therefore\ f(x) = -x(x-6)(x-6)$

$= -x(x^2 - 12x + 36)$

$= -x^3 + 12x^2 - 36x$

$a = 12;\quad b = -36;\quad c = 0$

ii) For point A

$f'(x) = 0$

$-3x^2 + 24x - 36 = 0$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

$x = 6$ or $x = 2$

At A $x = 2$ any $y = -8 + 48 - 72$

$= -32$

A(2; −32)

b) i) Co-ordinates of T (3; f (3))

$f(3) = -27 + 108 - 108 = -27$

T(3; −27)

ii) $f'(x) = -3x^2 + 24x - 36$

$f'(3) = -27 + 72 - 36$

$= 9$

Equation of tangent

$y + 27 = 9(x - 3)$

$\Rightarrow y = 9x - 54$

iii) Equation of normal $\quad (m = -\frac{1}{9})$

$y + 27 = -\frac{1}{9}(x - 3)$

$y + 27 = -\frac{x}{9} + \frac{1}{3}$

2. $y = a(x - 1)(x - 2)^2$

$-18 = a(2)(1)^2$

$a = -9$

$y = -9(x - 1)(x - 2)^2$

3. $\frac{dy}{dx} = -3x^2 + 2bx + c$

at $x = 1 \quad -3(1)^2 + 2b(1) + c = 0$

$-3 + 2b + c = 0$

$c = 3 - 2b$

(1; 0) is on the curve

$\therefore \quad 0 = -1 + b + c$

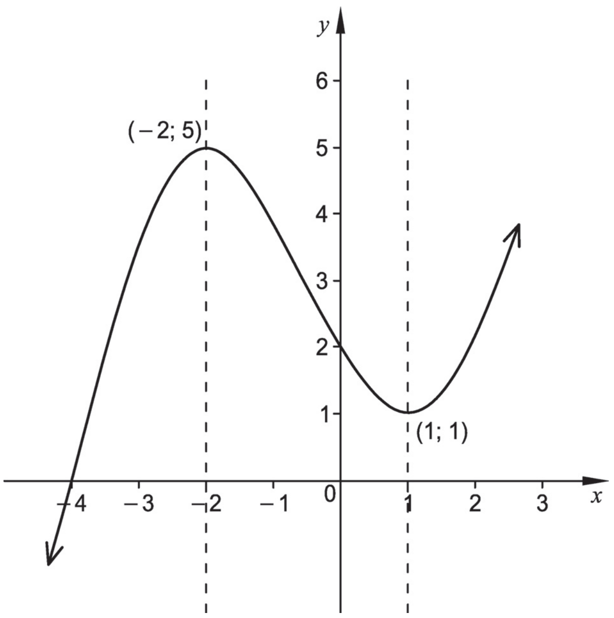
$c = 1 - b$

$c = 3 - 2b$

$0 = -2 + b$

$\therefore b = 2$

$c = -1$

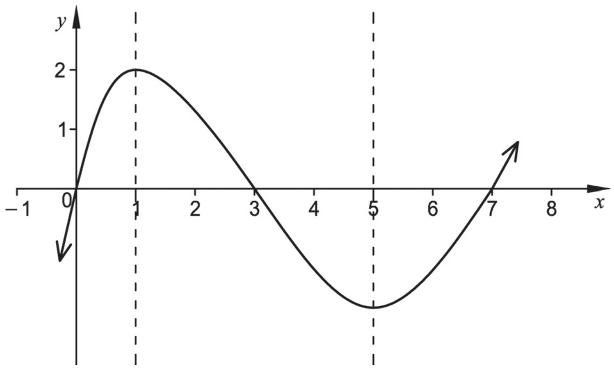


4.

5. a) 2 b) 6 by symmetry

c) [1; 5] d) $x = 1 \quad x = 5$

e) $f'(x)$



6. a) $f'(x) = 2x - 3x^2$

$f'\left(\frac{1}{3}\right) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$g'(x) = 1 - 2ax$

$g'\left(\frac{1}{3}\right) = 1 - \frac{2a}{3}$

$1 - \frac{2a}{3} = \frac{1}{3}$

$3 - 2a = 1$

$-2a = -2$

$a = 1$

b) $2x - 3x^2 = 1 - 2x$

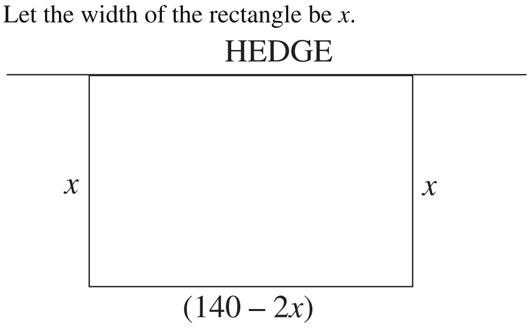
$3x^2 - 4x + 1 = 0$

$(3x - 1)(x - 1) = 0$

$x = 1$

Lesson 24

1. Let the width of the rectangle be x.



Then the length wiill be $(140 - 2x)$

Then the area: Area = length × breadth

$A(x) = (140 - 2x)x$ (We find an expression that relates what we want to work with.)

$= 140x - 2x^2$

This expression is a quadratic equation, which has a maximum turning point since $a < 0$.

To determine the maximum: differentiate the expression and solve it equal to zero.

Thus: $A(x) = 0$ to optimise

$\therefore 140 - 4x = 0$

$\therefore 4x = 140$

$\therefore x = 35 \rightarrow$ This is the dimensions of the rectangle for maximum A

Then: Maximum Area: $A(35) = 100(35) - 2(35)^2$

$= 3\,500 - 2\,450$

$= 1\,050$ m^2 is the maximum area

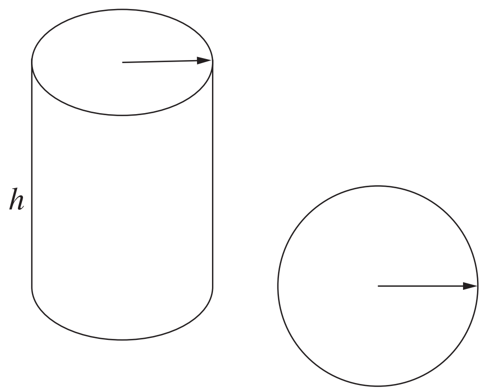
The Dimensions: Length $140 - 70 = 70$ m

Breadth 35 m

2. a) $V = x(2x)(180 - 3x)$

   b) 40 mm

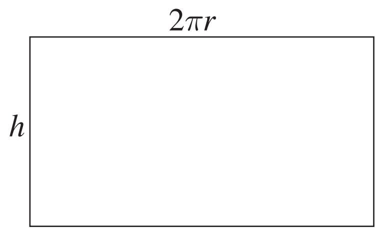
3. For a closed cylinder of radius r and height h, we open it up to determine its surface area.



For the circular part: There are two of them

Surface area of the top and bottom: $S_1 = \pi r^2$

For the walls:



Surface area : $S_2 = 2\pi rh$

Thus the total surface area: $S(r; h) = 2S_1 + S_2$
$= 2\pi r^2 + 2\pi rh$

Our problem now is that our surface area is in terms of two variables, so we cannot yet differentiate it in order to optimise the area. We need to get rid of either r or h, so that differentiation can take place as per normal. This is the reason why the volume has been supplied. We use the volume, where h is the dominant dimension taking it into the third dimension, and reduce it to two dimensions by removing the height from the area equation.

Thus: Volume = (Base Area) × (Height)

$2\,000\pi = (\pi r^2) \times (h)$

$\therefore h = \frac{2\,000\pi}{\pi r^2} = \frac{2\,000}{r^2}$ … (a)

Now: $S(r; h) = 2\pi r^2 + 2\pi rh$

$\therefore S(r) = 2\pi r^2 + 2\pi r \cdot \left(\frac{2\,000}{r^2}\right)$

$\therefore S(r) = 2\pi r^2 + \frac{4\,000x}{r}$ … (b)

$= 2\pi r^2 + 4\,000\pi r^{-1}$

To optimise: $S'(r) = 0$

$\therefore 4\pi r - \frac{4\,000}{x^2} = 0$

$\therefore 4\pi r^3 - 4\,000\pi = 0$

$\therefore r^3 = 1\,000$

$\therefore r = 10$

And now from (a): $h = \frac{2\,000\pi}{\pi r^2} = \frac{2\,000}{10^2} = \frac{2\,000}{100} = 20$

Thus the dimensions for minimum surface area: height – 20 cm
Radius – 10 cm

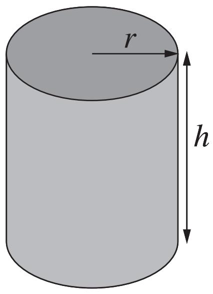
For the value of this minimum surface area: From (b) it follows that

$S(r) = 2\pi r^2 + \frac{4\,000x}{r}$

$\therefore S(10) = 2\pi(10)^2 + \frac{4\,000\pi}{10} = 200\pi + 400\pi = 600\pi \text{ cm}^2$.

Thus the minimum area is $S = 600\pi \text{ cm}^2$.

4. We let the height of the can be h cm and the radius be r cm

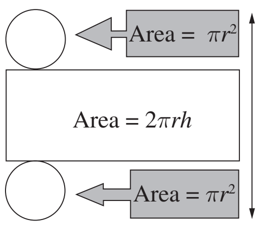


Then we can find an expression for the volume:

Volume = Base area × height

$\therefore 4\,000 = \pi r^2 h$

$\therefore h = \frac{4\,000}{\pi r^2}$



Area of the top and bottom:

$A(r; h) = 2\pi r^2$

Area of the walls:

$A(r; h) = 2\pi rh$

The cost involved in production:

Top : Walls = 2 : 1

$\frac{\text{Top}}{\text{Walls}} = \frac{2}{1}$

$\therefore$ Top = 2.Walls

Cost = 2(Circular area) + Wall area

$\therefore C(r; h) = 2(2\pi r^2) + 2\pi rh$

$\therefore C(r) = 4\pi r^2 + 2\pi r\left(\frac{4\,000}{\pi r^2}\right)$

$\therefore C(r) = 4\pi r^2 + \frac{8\,000}{r}$

To optimise the cost:

$C'(r) = 0$

$\therefore 8\pi r - \frac{8\,000}{r^2} = 0$

$\therefore 8\pi r^3 = 8\,000$

$\therefore \pi r^3 = 1\,000$

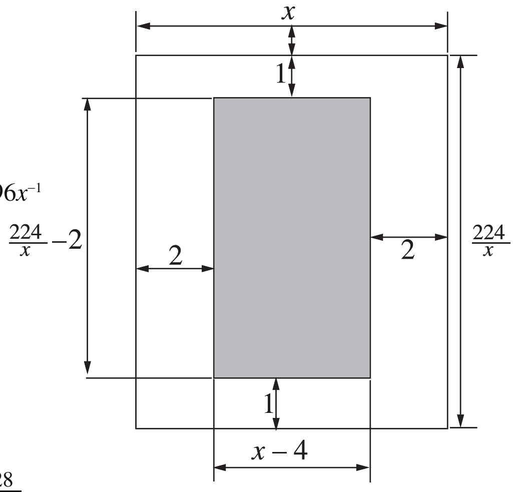
$\therefore r = \frac{10}{\sqrt[3]{\pi}}$

For the ratio: $h : r = 4 : 1$

5. Let the one side of the paper be x cm

Then the other side will be $\frac{224}{x}$ cm

The printing dimensions will be:



Thus: Printing area $= \left(\frac{224}{x} - 2\right)(x - 4)$

$\therefore P(x) = 224 - 2x - \frac{896}{x} + 8 = 232 - 2x - 896x^{-1}$

To optimise: $P'(x) = 0$

$\therefore -2 + 896x - 2 = 0$

$\therefore -2x^2 = -896$

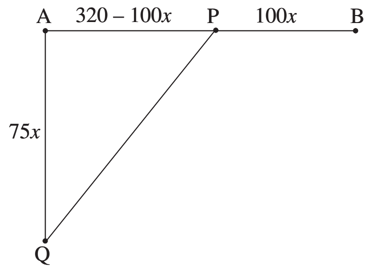
$\therefore x^2 = 448$

$\therefore x = 8\sqrt{7}$ since $x > 0$

Thus the length: $8\sqrt{7}$ and the width $\frac{224}{8\sqrt{7}} = \frac{28}{8\sqrt{7}}$

6. Assuming this happens after they have travelled for x hours, at point P:

Bus A travelled a distance of: $75x$ km



Bus B travelled a distance of: $100x$ km

The distance that they are apart:

$PQ = \sqrt{(320 - 100x)^2 + (75x)^2}$ (Pythagoras)

$= \sqrt{102\,400 - 64\,000x + 10\,000x^2 + 5\,625x^2}$

$= \sqrt{15\,625x^2 - 64\,000x + 102\,400}$

This is a quadratic equation under the surd, and we only have to find its turning point to find the minimum value:

Thus: $D(x) = 15\,625x^2 - 64\,000x$

To optimise: $D'(x) = 0$

$\therefore 31\,250x - 64\,000 = 0$

$\therefore x = \frac{64\,000}{31\,250}$

$\therefore x = 2{,}048$ hours

$\therefore x = 2$ hours 3 min

$\therefore x = 2$ hours (rounded to nearest hour)

The actual distance apart: $PQ = \sqrt{15\,625(2)^2 - 64\,000(2) + 102\,400}$

$PQ = 30\sqrt{41}$

$= 192$ km.

7. From Total Surface area:

$2\pi rh + \pi r^2 + 2\pi r^2 = 2\pi rh + 3\pi r^2 = 20\pi$.

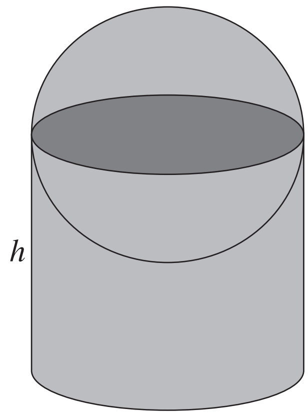
$\therefore h = \frac{20 - 3r^2}{2r}$

Volume $= \frac{2}{3}\pi r^3 + \pi r^2 h = \frac{2}{3}\pi r^3 + \pi r^2\left(\frac{20 - 3\pi}{2r}\right)$

$= \frac{2}{3}\pi r^3 + 10\pi r - \frac{3}{2}\pi r^3$

$= 10\pi r + \frac{4 - 9}{6}\pi r^3$

$= 10\pi r - \frac{5}{6}\pi r^3$



$V'(r) = 10\pi - 5\pi r^2 = 0$

$\therefore 5\pi r^2 = 10\pi$

$\therefore r^2 = 4$

$\therefore r = 2$

Then Max Volume $= 10\pi(2) - \frac{5}{6}\pi(8) = \frac{40\pi}{3}$

8. $3y = 12 - x^2$ and $6y = x^2 - 12$.
$\therefore\ y = 4 - \frac{1}{3}x^2$ and $y = \frac{1}{6}x^2 - 2$
PQ = $2x$ and QR = $4 - \frac{1}{3}x^2 - \frac{1}{6}x^2 + 2 = 6 - \frac{1}{2}x^2$
$\therefore$ Area = $2x\left(6 - \frac{1}{2}x^2\right) = 12x - x^3$
$\therefore\ A'(x) = 12 - 3x^2 = 0$
$\therefore\ x^2 = 4$
$\therefore\ x = 2$
So PQ = 4 and RQ = 4
Thus Maximum Area = $16u^2$9. $x = 10$ mm

10. 2 250 cm^3
11. $x = 20$ $y = 10$
12. 130 by 130

Lesson 25

1. 45 m; 21 $m{\cdot}s^{-1}$; deceleration of 2 $m{\cdot}s^{-2}$
2. a) 56 m b) 9 seconds c) -36 $m{\cdot}s^{-1}$ d) $4\frac{1}{2}$ seconds
e) 81 m f) -4 $m{\cdot}s^{-2}$
3. a) $s = \frac{1}{3}t^3 - \frac{1}{2}t^2 + 2t + c$
b) $3 = c$ c) $a = 2t - 1$
$\therefore\ s = \frac{1}{3}t^3 - \frac{1}{2}t^2 + 2t + 3$
$s = \frac{1}{3}(6)^3 - \frac{1}{2}(6)^2 + 12 + 3$
$s = 60$
4. a) 30 $m{\cdot}s^{-1}$ b) 10 seconds c) 200 m
5. a) 6 m^3 per minute b) 5 minutes c) 30 m^3
6. a) 675 u per day b) 15 days
7. a) Decreasing by 26 cm^3 second
(-26 cm^3 second)
b) 28 cm^3 second
8. a) 3 m b) Decreasing at $3\frac{3}{4}$ m per hour
c) At 13:00 d) At 10:20

TIPS FOR THE TEACHER

Lesson 18

- This section counts for a lot of marks so it is important that the learners practise many examples.
- Always stress that the derivative is the gradient at a point.
- Try to bring in practical examples.
- Be careful of the notation.

Lesson 19

- This section counts for a lot of marks so it is important that the learners practise many examples.
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Lesson 20

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Lesson 21

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- Always stress that the derivative is the gradient at a point.
- Try to bring in practical examples.
- Be careful of the notation.

Lesson 22

- This section counts for a lot of marks so it is important that the learners practise many examples.
- Always stress that the derivative is the gradient at a point.
- Try to bring in practical examples.
- Be careful of the notation.

Lesson 23

- Keep stressing to the learners that the derivative is the gradient.
- It is a good idea to make them do rough sketches of graphs.

Lesson 24

- Encourage the learners to draw pictures before they make equations.
- It is a good idea to make the first equation the equation they need to differentiate.

Lesson 25

- Stress the difference between increasing and decreasing functions.
- Rate is a gradient so whenever they see the word rate they must differentiate.