MATHEMATICAL MODELLING

Learning Outcomes and Assessment Standards

Learning Outcome 2: Functions and algebra
Assessment Standard

- Use mathematical models to investigate problems that arise in real-life contexts.
- Making conjecture, demonstrating and explaining their validity.
- Expressing and justifying mathematical generalisations of situations.
- Using the various representations to interpolate and extrapolate.

Overview

In this lesson you will:

- Solve the normal routine word sum problems.
- Look at non-routine questions involving proof.

For these problems we usually:

(1) State our variable.
(2) Organise the information in a table form if possible.
(3) Recognise our mathematical tools that apply in the organising equation.
(4) Solve our problem.
(5) Check our answer for validity.

Lesson

Routine problems

1. **Area:** The area of a room is 20 m^2.

If the length is increased by 3 m and the width by 1 m, the room will double in area. Determine the original dimensions of the room.

Example 1

Farmer Phillip has 1 500 trays of tomatoes which he can sell to the market today at R20 per tray. His neighbour informs him that he should wait a few days, as the market price will increase by about R2 per day. He knows that if he does this, he will lose about 5 trays of tomatoes per day due to them deteriorating. He asks you to help him by claculating if it will be a good idea to wait a few days longer.

Draw up a detailed explanation for him.

Solution

(1) Define the variable you will use:

Your choice of variables is very important. The longer he waits, the price increases, but the amount he can sell becomes less. So:

Let x be the number of days that he waits.

Then: Amount he has to sell after x days: $(1\,500 - 5x)$ trays

- 1 500: the amount of trays he started with
- $5x$: he loses 5 trays for each day he waits

His income per tray will be: $(20 + 2x)$

price he could get today ← $(20 + 2x)$ → the price that increase with R2 per tray per day

His total income = Units sold × Price per unit

$= (1\,500 - 5x)(20 + 2x)$

$I(x) = 30\,000 + 2\,900x - 10x^2$

So we can see that this is a quadratic equation, for which we can complete the square:

$-10\,(x^2 - 290x - 3\,000)$

$= -10\,(x^2 - 290x + (145)^2 - (145)^2 - 3\,000)$

$= -10[(x - 145)^2 - 24\,025]$

$= -10(x - 145)^2 + 240\,250$

So he has to wait 145 days, and then sell to the market to get an income of R240 250.

If he sells today, he will make R20 × (1 500) = R30 000

If he waits for 45 days:

He has (1 500 – 5(145)) trays = 775 trays

He can sell at (20 + 2(145)) = R310 per tray)

So his income will be R240 250

Example 2: Rate

Two machines, working together, complete a job in 2 hrs 24 min. Working on its own the one machine would take 2 hours longer than the other. How long does the slower machine take?

⇒ Let the slower machine take x hours

Then the faster machine will take $x - 2$ hours.

Together: 2,4 hours

$\therefore$ In 1 hour:

$\frac{1}{x} + \frac{1}{x-2} = \frac{1}{2{,}4}$

$\therefore (x - 2)2{,}4 + (x)2{,}4 = x(x - 2)$ … (multiply by LCD)

$\therefore 2{,}4x - 4{,}8 + 2{,}4x = x^2 - 2x$

$\therefore 4{,}8x - 4{,}8 = x^2 - 2x$

$\therefore 48x - 48 = 10x^2 - 20x$ … (multiply by 10 to rid decimals)

$\therefore 10x^2 - 68x + 48 = 0$

$\therefore 5x^2 - 34x + 24 = 0$

$\therefore (5x - 4)(x - 6) = 0$

$\therefore 5x = 4$ or $x = 6$

$x = \frac{4}{5}$ hours

impossible

Thus the slower machine takes 6 hours.

Example 3: Speed

A passenger train travels 10 km/h faster than a goods train. They have to travel from Station A to B, 100 km apart. It takes the goods 30 minutes longer than the passenger train to cover the distance. What is the speed of the goods train?

$\Rightarrow$ We want the speed of the goods train:

Let the speed be x:

	D	S	T
Goods	100	x	$\frac{100}{x}$
Passenger	100	$x + 10$	$\frac{100}{x+10}$

D = Speed × Time

$\therefore t = \frac{D}{S}$

The determing factor here is time:-

$T_{GOODS} = T_{PASS} + 30$ min

$\therefore \frac{100}{x} = \frac{100}{x+10} + 0{,}5$ (NB Speed in km/h, so time must be in hours)

$\therefore 100(x + 10) = 100x + \frac{1}{2}x(x + 10)$

$\therefore 200(x + 10) = 200x + x(x + 10)$

$\therefore 200x + 2\,000 = 200x + x^2 + 10x$

$\therefore x^2 + 10x - 2\,000 = 0$

$\therefore (x - 40)(x + 50) = 0$

$\therefore x = 40$ or $x = -50$

n.a.

Thus the speed of the goods train is 40 km/h.

Example 4: Financial

A person sets aside R1 800 for his holiday and budgets x rands per day. However, he spends R10 more per day and finds that he has to reduce his stay by 2 days.

3.1 Write down 2 expressions in x representing the number of days he actually spends on holiday.

3.2 Determine his budget expenditure per day whilst on holiday.

$\Rightarrow$ Total: R1 800 at Rx per day.

Now: $(x + 10)$ per day.

3.1 Planned days on holiday: $\frac{1\,800}{x}$

Actual days on holiday: $\frac{1\,800}{x+10}$

$\therefore \frac{1\,800}{x+10} = \frac{1\,800}{x} - 2$

Thus either $\frac{1\,800}{x+10}$ or $\frac{1\,800}{x} - 2$

3.2 $\frac{1\,800}{x+10} = \frac{1\,800}{x} - 2$

$\therefore 1\,800x = 1\,800(x + 10) - 2x(x + 10)$

$\therefore 1\,800x = 1\,800x + 18\,000 - 2x^2 - 20x$

$\therefore 2x^2 + 20x - 18\,000 = 0$

$\therefore x^2 + 10x - 9\,000 = 0$

$\therefore (x + 100)(x - 90) = 0$

$\therefore x = -100$ or $x = 90$

n.a.

$\therefore$ Budget expenditure per day – R90,00

Now do Activities 1 to 7

Non-routine problems and conjectures

- Select any three-digit number with all digits different from one another.
- Write down all possible two-digit numbers that can be formed from the three-digit number you chose.
- Divide the sum of the digit
- What is your answer.

Conjecturing and Proof

Do the following:

(1) Select a three digit number, with all digits different from one another and no digit equal to zero.

(2) Write down all the possible two digit numbers that can be formed from the three digits in your chosen number.

(3) Now divide the sum of the two digit numbers you formed, by the sum of the digits in your orginal three digit number. Write the answer down.

(4) Now repeat the first three steps for two more three digit numbers, and write the answers down.

(5) What do you notice?

(6) Formulate a conjecture from your observation.

(7) Now use algebraic methods to prove your conjecture true.

(8) If there is a zero digit, what will the quotient always be?

Steps (1 – 4)

$\Rightarrow$ The number: 123

Six possibilities: 12; 13; 23; 32; 31; 21

Sums: $\frac{12 + 13 + 23 + 32 + 31 + 21}{1 + 2 + 3} = \frac{132}{6} = 22$

The number: 479

Six possibilities: 47; 49; 79; 97; 94; 74

Sums: $\frac{47 + 49 + 79 + 97 + 94 + 74}{4 + 7 + 9} = \frac{440}{20} = 22$

The number: 659

Six possibilities: 65; 59; 69; 95; 56; 96

Sums: $\frac{440}{6+5+9} = \frac{440}{20} = 22$

(5) You will notice that your solution is always 22.

(6) If we take a three digit number and calculate the sum of all the two digit numbers that can be formed by using the digits in the three digit number, and we divide this sum by the sum of the digits in the 3 digit number, the result will always be 22.

(7) Proof

Let the three digit number be *abc*. According to our base 10 place value system, this means that the number is:

$abc \rightarrow a(100) + b(10) + c = 100a + 10b + c$

The two digit numbers will be:

$10a + b; 10a + c; 10b + c; 10c + b; 10c + a; 10b + a$

The sums:

$$\frac{10a + b + 10a + c + 10b + c + 10c + b + 10c + a + 10b + a}{a + b + c}$$

$$= \frac{20a + 2a + 20b + 2b + 20c + 2c}{a + b + c}$$

$$= \frac{22a + 22b + 22c}{a + b + c}$$

$$= \frac{22(a + b + c)}{a + b + c}$$

$= 22$

(8) $a\,0\,b \Rightarrow 100a + 0 + b$

$10a + 0; 10a + b; 10b + 0; 10b + a$

Sums: $\frac{10a + 0 + 10a + b + 10b + 0 + 10b + a}{a + b}$

$$= \frac{20a + a + 20b + b}{a + b}$$

$$= \frac{21a + 21b}{a + b}$$

$= 21$

Activity 1

1. A printed rectangular card is to have a perimeter of 42 cm. The printed area is to be 50 cm^2. The margins above and below in the length of the card are to be 1 cm. The side margins are each to be 2 cm. Find the dimensions of the card.

2. The difference between two natural numbers is 3 and their product is 40. Calculate the numbers.

3. If the speed of a new locomotive was 10 km/h faster than an older model, then the journey between two stations 100 km apart would be shortened by 30 minutes. Determine the speed of the old model.

4. An electrician and his apprentice, working together, complete the wiring of a building in 20 days. If the apprentice were to do the work on his own, he would take 9 days longer than the electrician. How long would it take each one of them if they did the work on their own.

5. A boat can travel 30 km/h in still water. The boat takes one hour to travel 12 km upstream and back. Find the speed at which the stream is flowing.

6. A train which is x km long normally takes $\frac{5x}{2}$ minutes to pass through a tunnel which is 1 500 m long. Due to heavy rains it is forced to reduce its normal speed by 20 km/h and takes $\frac{15x}{4}$ minutes to pass through the tunnel. Find the length of the train.

7. The denominator of a positive fraction is one more than the square of the numerator. If the numerator is increased by 1 and the denominator decreased by 3, the value of the new fraction is $\frac{1}{4}$. Find the original fraction.

8. a) A group of people, consisting of x persons, must share an amount of R120 equally. How much must each person pay. (in terms of x)?

   b) Four of these persons refuse to pay. How much must the rest each pay to square the amount (in terms of x)?

   c) If the remainder of the group (i.e. those who are prepared to pay) must each pay R5 more than their original share, calculate how many people there were in the original group.

9. A man bought a number of sheep for R960. He lost four, but by selling the rest at a profit of R8 a piece, he made good his loss. How many sheep did he buy?